A proof of Shelah's eventual categoricity conjecture in universal classes[1]

Sebastien Vasey

Carnegie Mellon University

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University of Leeds, UK

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Introduction

Observation

Let λ be an uncountable cardinal.

- There is a unique $\mathbb{Q}$-vector space with cardinality λ.
- There is a unique algebraically closed field of characteristic zero with cardinality λ.

Definition (Łoś, 1954)

A class K of structure is *categorical in* λ if it has exactly one model of cardinality λ (up to isomorphism).

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Question

If K is "reasonnable", can we say something about the class of cardinals in which K is categorical?

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Theorem (Morley, 1965)

Let K be the class of models of a countable first-order theory. If K is categorical in *some* $\lambda \geq \aleph_1$, then K is categorical in *all* $\lambda' \geq \aleph_1$.

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Conjecture (Shelah, 197?)

Let K be the class of models of an $\mathbb{L}_{\omega_1,\omega}$-sentence. If K is categorical in *some* $\lambda \geq \beth_{\omega_1}$, then K is categorical in *all* $\lambda' \geq \beth_{\omega_1}$.

Main result

Definition

An $\mathbb{L}_{\omega_1,\omega}$-sentence is *universal* if it is of the form $\forall x_0 \forall x_1 \ldots \forall x_n \psi$, with ψ quantifier-free.

Theorem (V.)

Let K be the class of models of a *universal* $\mathbb{L}_{\omega_1,\omega}$-sentence. If K is categorical in *some* $\lambda \geq \beth_{\beth_{\omega_1}}$, then K is categorical in *all* $\lambda' \geq \beth_{\beth_{\omega_1}}$.

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A class K of structures in a fixed vocabulary $\tau(K)$ is *universal* if it is closed under isomorphisms, substructure, and union of $\subseteq$-increasing chains.

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Theorem (V.)

Let K be a universal class. If K is categorical in *some* $\lambda \geq \beth_{\left(2^{|\tau(K)|+\aleph_0}\right)^+}$, then K is categorical in *all* $\lambda' \geq \beth_{\left(2^{|\tau(K)|+\aleph_0}\right)^+}$.

A step back: abstract elementary classes

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Definition (Shelah, 1985)

An *abstract elementary class (AEC)* is a partial order $\mathbf{K} = (K, \leq_{\mathbf{K}})$ where K is a class of structures in a fixed vocabulary $\tau(\mathbf{K})$, and:

1. K is closed under isomorphism, $\leq_{\mathbf{K}}$ respects isomorphisms.
2. If $M \leq_{\mathbf{K}} N$, then $M \subseteq N$.
3. Coherence: If $M_0 \subseteq M_1 \leq_{\mathbf{K}} M_2$ and $M_0 \leq_{\mathbf{K}} M_2$, then $M_0 \leq_{\mathbf{K}} M_1$.
4. Downward Löwenheim-Skolem-Tarski axiom: There is a least cardinal $\text{LS}(\mathbf{K}) \geq |\tau(\mathbf{K})| + \aleph_0$ such that for any $N \in K$ and $A \subseteq |N|$, there exists $M \leq_{\mathbf{K}} N$ containing A of size at most $\text{LS}(\mathbf{K}) + |A|$.
5. Chain axioms: If δ is a limit ordinal, $\langle M_i : i < \delta \rangle$ is a $\leq_{\mathbf{K}}$-increasing chain in K, then $M_\delta := \bigcup_{i<\delta} M_i$ is in K, and:
   5.1 $M_i \leq_{\mathbf{K}} M_\delta$ for all $i < \delta$.
   5.2 If $N \in K$ is such that $M_i \leq_{\mathbf{K}} N$ for all $i < \delta$, then $M_\delta \leq_{\mathbf{K}} N$.

Examples

- If K is a universal class, then $\mathbf{K} = (K, \subseteq)$ is an AEC with $\mathrm{LS}(\mathbf{K}) = |\tau(K)| + \aleph_0$.

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- For $\psi \in \mathbb{L}_{\omega_1,\omega}$, Φ a countable fragment containing ψ, $\mathbf{K} := (\text{Mod}(\psi), \preceq_\Phi)$ is an AEC with $\text{LS}(\mathbf{K}) = \aleph_0$.

Shelah's eventual categoricity conjecture for AECs

An AEC that is categorical in *some* high-enough cardinal is categorical in *all* high-enough cardinals.

Some earlier approximations

Theorem

- (Shelah 1999, Grossberg-VanDieren 2006) Any tame AEC with amalgamation that is categorical in *some* high-enough successor cardinal is categorical in *all* high-enough cardinals.
- (Shelah 2009; assuming an unpublished claim) Assume $2^{\lambda} < 2^{\lambda^+}$ for all cardinals λ. Any AEC with amalgamation that is categorical in *some* high-enough cardinal is categorical in *all* high-enough cardinals.

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Theorem (Makkai-Shelah 1990, Kolman-Shelah 1996, Boney 2014)

Tameness can be derived from
a proper class of strongly compact cardinals and amalgamation from (categoricity and) a proper class of measurable cardinals.

Advantages

Theorem (V.)

If a universal class $\mathbf{K}$ is categorical in *some* $\lambda \geq \beth_{\beth_{\left(2^{\mathrm{LS}(\mathbf{K})}\right)^+}}$, then $\mathbf{K}$ is categorical in *all* $\lambda' \geq \beth_{\beth_{\left(2^{\mathrm{LS}(\mathbf{K})}\right)^+}}$.

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We *do* assume that **K** is a universal class. But the proof also applies to AECs satisfying more general hypotheses.

Two main steps of the proof

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Proof steps.

Write $h(\chi) := \beth_{(2^\chi)^+}$.

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2. $\mathbf{K}^*$ has amalgamation, is $\mathrm{LS}(\mathbf{K}^*)$-tame, and has primes over sets of the form $M \cup \{a\}$.

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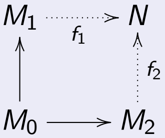
Step 2: For any such $\mathbf{K}^*$, categoricity in *some* $\mu \geq h(\mathrm{LS}(\mathbf{K}^*))$ implies categoricity in *all* $\mu' \geq h(\mathrm{LS}(\mathbf{K}^*))$. □

Amalgamation

Definition

An AEC $\mathbf{K}$ has *amalgamation* if whenever $M_0 \leq_{\mathbf{K}} M_\ell$, $\ell = 1, 2$, there exists $N \in \mathbf{K}$ and $f_\ell : M_\ell \xrightarrow[M_0]{} N$.



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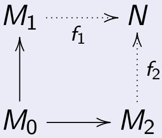


Amalgamation can fail in general AECs, even in universal classes.

Theorem (Kolesnikov and Lambie-Hanson, 2015)

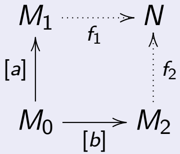
For every $\alpha < \omega_1$, there exists a universal class in a countable vocabulary that has amalgamation up to $\beth_\alpha$ but fails amalgamation starting at $\beth_{\omega_1}$.

Galois types and tameness

Definition

For **K** an AEC with amalgamation:

- (Shelah) $\mathrm{gtp}(a/M_0; M_1) = \mathrm{gtp}(b/M_0; M_2)$ if there exists N with:



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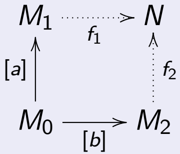


  and $f_1(a) = f_2(b)$.

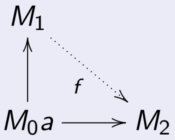
- (Grossberg-VanDieren) **K** is χ-*tame* if whenever $\mathrm{gtp}(a/M_0; M_1) \neq \mathrm{gtp}(b/M_0; M_2)$, there exists $N \leq_{\mathbf{K}} M_0$ with $\|N\| \leq \chi$ and $\mathrm{gtp}(a/N; M_1) \neq \mathrm{gtp}(b/N; M_2)$.

Primes

Definition (Shelah)

An AEC **K** *has primes* if for any Galois type p over M_0, there exists a triple (a, M_0, M_1) such that $p = \mathrm{gtp}(a/M_0; M_1)$ and whenever $p = \mathrm{gtp}(b/M_0; M_2)$, there exists $f : M_1 \underset{M_0}{\longrightarrow} M_2$ with $f(a) = b$.

(in the diagram below, $a = b$):



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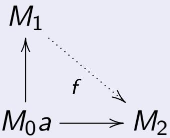

In vector spaces, the span of $M_0 a$ gives a prime model over $M_0 a$. More generally, in universal classes the closure of $M_0 a$ to a substructure gives a prime model over $M_0 a$.

Proof sketch for a weak version of step 2

Let **K** be a LS(**K**)-tame AEC with amalgamation and primes. Let $\mu < \lambda$ both be "high-enough" categoricity cardinals. We show that **K** is categorical in μ^+.

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4. Goodness transfers up (uses tameness and primes): $\mathbf{K}_{\neg p}$ is "good" also above μ.
5. By "goodness", $\mathbf{K}_{\neg p}$ has a model of cardinality λ.
6. This contradicts categoricity in λ (the model there is saturated).

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- John T. Baldwin, *Categoricity*. University Lecture Series, vol. 50, American Mathematical Society, 2009.
- Will Boney and Sebastien Vasey, *A survey on tame abstract elementary classes*. To appear in Beyond first order model theory.