Categoricity and multidimensional diagrams

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Accessible categories and their connections
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Method of multidimensional diagrams: the short version

Idea: in an accessible category, if we understand the small objects well, we understand everything.

Pseudo-definition: a (nice) category is *excellent* if any n-dimensional cube without top corner consisting of *small* objects can be "nicely amalgamated".

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Two key themes/pseudo-results:

1. If the category is excellent, then we understand it well. In particular, "small" in the definition above can be removed.
2. If any *low-dimensional* cube of objects can be nicely amalgamated (i.e. we have "enough base cases"), then the category is excellent.

The plan

- The categoricity spectrum problem.
- Excellence: how it works.
- Excellence: how to derive it.
- Some questions.

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Example

- The class of all groups is not categorical in any cardinal.
- The class of all sets is categorical in all infinite cardinals.
- The class of $\mathbb{Q}$-vector spaces is categorical in all uncountable cardinals, not in $\aleph_0$.
- The class of dense linear orders without endpoints is categorical only in $\aleph_0$.

These are the only categoricity spectrums, for classes axiomatized by a countable first-order theory.

Categoricity (2)

Theorem (Morley, 1965)

A countable first-order theory categorical in *some* uncountable cardinal is countable in *all* uncountable cardinals.

The methods used for the proof of Morley's theorem have played a big role in model theory and beyond. Can this be generalized to other setups?

Abstract elementary classes (Shelah, 1980s)

An AEC is a pair $\mathbf{K} = (K, \leq_{\mathbf{K}})$, where K is a class of structures in a fixed vocabulary $\tau(\mathbf{K})$ and $\leq_{\mathbf{K}}$ is a partial order on $\mathbf{K}$ satisfying some of the basic category-theoretic properties of $(\mathrm{Mod}(T), \preceq)$.

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For example, $\mathbf{K}$ is closed under unions of $\leq_{\mathbf{K}}$-increasing chains and satisfies the downward Löwenheim-Skolem-Tarski theorem. More precisely, there exists a (least) cardinal $\mathrm{LS}(\mathbf{K}) \geq |\tau(\mathbf{K})| + \aleph_0$ such that for any $M \in \mathbf{K}$ and any $A \subseteq |M|$, there is $M_0 \leq_{\mathbf{K}} M$ containing A with $\|M_0\| \leq |A| + \mathrm{LS}(\mathbf{K})$.

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Examples include $(\mathrm{Mod}(T), \preceq)$ (where $\mathrm{LS}(\mathbf{K}) = |T|$), $(\mathrm{Mod}(\psi), \preceq_\Phi)$ (where $\mathrm{LS}(\mathbf{K}) = |\Phi| + |\tau(\Phi)| + \aleph_0$), $\psi \in \mathbb{L}_{\infty,\omega}$, and more generally classes of models of $\mathbb{L}_{\infty,\omega}(\exists^{\geq\lambda})$ sentences.

Fact (Beke-Rosický, 2012)

Any finitely accessible category with all morphisms mono is an AEC, and any AEC is an accessible category with directed colimits and all morphisms mono.

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The big open problem is:

Conjecture (Shelah's *eventual* categoricity conjecture for AECs)

The categoricity spectrum of an AEC is either bounded or contains an end segment.

Categoricity at a successor

Historically, most approximations to Shelah's eventual categoricity conjecture have assumed categoricity in a *successor* cardinal.

Why? Because if we are categorical in μ^+, we can just work in μ. If we are categorical in a limit μ, we have to work with models of different sizes below μ and it is much harder.

Theorem (Makkai-Shelah 1990, Grossberg-VanDieren 2006, Boney 2014)

If there is a proper class of strongly compact cardinals, then Shelah's eventual categoricity conjecture holds when we assume categoricity in a high-enough *successor* cardinal.

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Assume $2^{\mu} < 2^{\lambda}$ whenever $\mu < \lambda$. Then Shelah's eventual categoricity conjecture holds in AECs with the amalgamation property.

It is known (Kolman-Shelah, 1996) that categoricity above a *measurable* cardinal implies amalgamation below the categoricity cardinal. Thus we obtain that Shelah's eventual categoricity conjecture is *consistent* with a proper class of measurable cardinals.

The main technical result

Theorem (Shelah-V.)

1. Assume $2^\lambda < 2^\mu$ whenever $\lambda < \mu$. If an AEC has a superstable independence notion, then it is excellent.
2. Shelah's eventual categoricity conjecture holds for excellent AECs.

The cardinal arithmetic assumption can be bypassed in the presence of large cardinals.

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In the two setups mentioned above, it was known that the existence of a superstable independence notion followed from categoricity. The second part was also essentially known (all that was needed was a working definition of "excellent").

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So really, the main contribution is the first part: "two-dimensional nice amalgamation implies *n*-dimensional nice amalgamation for all *n*".

The categoricity spectrum in AECs with amalgamation

One can work harder and build superstable independence notions (i.e. prove the "base cases") with weaker hypotheses. This leads to a full characterization of the categoricity spectrum in AECs with amalgamation:

Theorem (V.)

Assume $2^\mu < 2^\lambda$ whenever $\mu < \lambda$. Let $\mathbf{K}$ be an AEC with amalgamation and arbitrarily large models. Let C be the categoricity spectrum of $\mathbf{K}$. Exactly one of the following holds:

1. $C = \emptyset$.
2. $C = [\text{LS}(\mathbf{K})^{+m}, \text{LS}(\mathbf{K})^{+n}]$ for some $m \leq n < \omega$.
3. $C = [\chi, \infty)$ for some $\chi < \beth_{\left(2^{\text{LS}(\mathbf{K})}\right)^+}$.

There are examples for all three cases.

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Note that an AEC always has no maximal models above a measurable cardinal (Boney 2014): one can take an sufficiently complete ultraproduct.

Excellence: the starting point

Theorem (Shelah, 1983)

Assume $2^{\aleph_n} < 2^{\aleph_{n+1}}$ for every $n < \omega$. Let ψ be an $\mathbb{L}_{\omega_1,\omega}$ sentence. If ψ is categorical in $\aleph_n$ for each $n < \omega$, then ψ is categorical in all infinite cardinals.

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It is really necessary in Shelah's theorem to make assumptions on all the $\aleph_n$'s: for each $n < \omega$, there is an example of Hart-Shelah (analyzed further by Baldwin-Kolesnikov) categorical in $\aleph_0, \ldots, \aleph_n$ but *not* in any $\lambda > \aleph_n$.

Getting existence

Let's work in an AEC $\mathbf{K}$ with $\mathrm{LS}(\mathbf{K}) = \aleph_0$ (for simplicity).

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$$M \dashrightarrow N$$

Proof.

$$M_0$$

□

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$$\begin{array}{c} M_1 \\ \uparrow \\ M_0 \end{array}$$

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$$\begin{array}{c} \cdots \\ \uparrow \\ M_1 \\ \uparrow \\ M_0 \end{array}$$

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$$\begin{array}{c} \bigcup_{i<\aleph_1} M_i \\ \uparrow \\ \cdots \\ \uparrow \\ M_i \\ \uparrow \\ \cdots \\ \uparrow \\ M_1 \\ \uparrow \\ M_0 \end{array}$$

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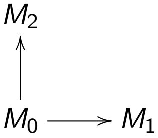

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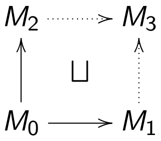

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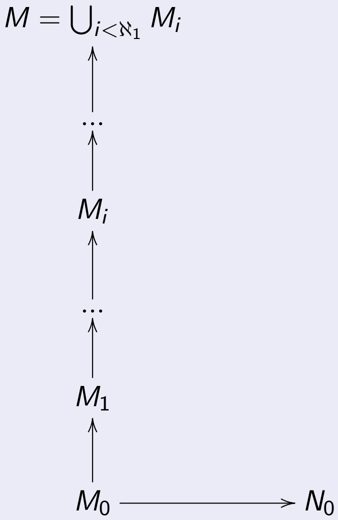
$$
\begin{array}{c}
M = \bigcup_{i<\aleph_1} M_i \\
\uparrow \\
\cdots \\
\uparrow \\
M_i \\
\uparrow \\
\cdots \\
\uparrow \\
M_1 \\
\uparrow \\
M_0
\end{array}
$$

□

Proof.



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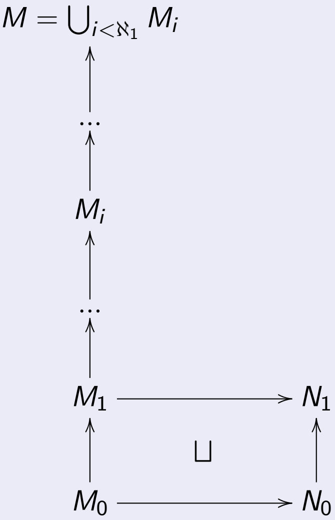

$\square$

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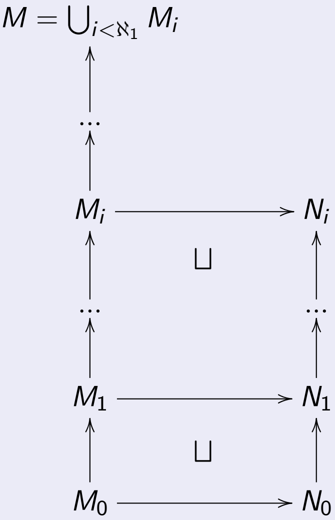

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$$
\begin{array}{ccc}
M = \bigcup_{i<\aleph_1} M_i & \longrightarrow & \bigcup_{i<\aleph_1} N_i \\
\uparrow & \sqcup & \uparrow \\
\cdots & & \cdots \\
\uparrow & & \uparrow \\
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Definition

For I a downward-closed system of sets, a (λ, I)-*system* is a sequence $\langle M_i : i \in I \rangle$ of models in $\mathbf{K}$ of cardinality λ such that for $u \subseteq v$ in I, $M_u \leq_{\mathbf{K}} M_v$. In other words, it is a functor from $(I, \subseteq)$ into $\mathbf{K}_\lambda$.

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An I-system is *disjoint* if $M_u \cap M_v = M_{u \cap v}$ for every $u, v \in I$. A system is called *strict* if $M_u \lneq_{\mathbf{K}} M_v$ for $u \subsetneq v$.

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We always identify a natural number n with $\{0, \ldots, n-1\}$. We write $\mathcal{P}^-(n)$ for $\mathcal{P}(n) \backslash \{n\}$ (think of it as an n-dimensional cube without the topmost corner).

The existence properties (2)

We talk about (λ, n)-systems instead $(\lambda, \mathcal{P}(n))$-system. Similarly, (λ, n^-)-systems are $(\lambda, \mathcal{P}^-(n))$-systems.

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We say $\mathbf{K}$ has (λ, n)-*extension* if any strict disjoint (λ, n^-)-system can be completed to a *strict disjoint* (λ, n)-system.

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Base cases:

- $(\lambda, 0)$ extension says that **K** has a model of cardinality λ.
- $(\lambda, 1)$-extension says that every model of cardinality λ can be properly extended.
- $(\lambda, 2)$-extension is disjoint amalgamation.

We showed before that $(\aleph_0, \leq 1)$-extension implies $(\aleph_1, 0)$-extension, and $(\aleph_0, \leq 2)$-extension implies $(\aleph_1, \leq 1)$-extension. This can be generalized to:

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1. **Q:** How do we prove the $(\aleph_0, < \omega)$-extension property? **A:** More on this later!
2. **Q:** What about *uniqueness* of the model in λ?

The uniqueness properties: the two-dimensional case

Naively, one may want to require any two amalgams to be unique up to isomorphism. This does not quite work. For example:

Example

Consider countably infinite sets B, C, D, E with $D = B \cup C \subseteq E$ and $E \backslash D$ infinite. Then D and E are not isomorphic over $B \cup C$.

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We also do not necessarily have pushouts or prime models. It is something we want to prove rather than assume.

Definition

Consider a span:

$$\begin{array}{ccc} M_1 & & \\ {\scriptstyle f_1}\uparrow & & \\ M_0 & \xrightarrow[f_2]{} & M_2 \end{array}$$

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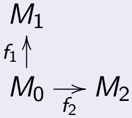
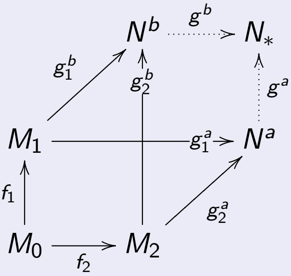


Two amalgams

$g_1^a : M_1 \to N^a, g_2^a : M_2 \to N^a$, $g_1^b : M_1 \to N^b, g_2^b : M_2 \to N^b$ of (f_1, f_2) are *equivalent* if there exists N and g^a, g^b making the following diagram commute:



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In the category of sets, any two disjoint amalgams of the same span are equivalent. This is not always the case.

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In the category of sets, any two disjoint amalgams of the same span are equivalent. This is not always the case.

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Let **K** be the class of graphs. One can amalgamate two graphs disjointly with all cross edges, or disjointly with no cross edges. These form non-equivalent amalgams.

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There are even examples in the class of algebraically closed fields of characteristic zero (categorical in every uncountable cardinal).
Solution: restrict oneself to only certain squares (called the *independent (or nonforking) squares*).

Stable independence

Definition (Lieberman-Rosický-V.)

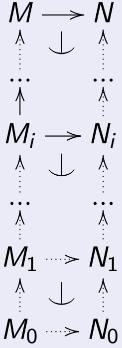
A *stable independence notion* is a class of squares (called *independent squares*) such that:

1. Independent squares are closed under isomorphisms and equivalence of amalgam.
2. Existence: any span can be amalgamated to an independent square.
3. Uniqueness: any two *independent* amalgam of the same span are equivalent.
4. Transitivity:

$$\begin{array}{ccccc} M_1 & \to & M_3 & \to & M_5 \\ \uparrow & \perp & \uparrow & \perp & \uparrow \\ M_0 & \to & M_2 & \to & M_4 \end{array} \Rightarrow \begin{array}{ccc} M_1 & \to & M_5 \\ \uparrow & \perp & \uparrow \\ M_0 & \to & M_4 \end{array}$$

Definition (stable independence notion - continued)

5. Symmetry: "swapping the ears" M_1 and M_2 preserves independence.
6. Accessibility: the arrow category whose morphisms are independent squares is accessible. This implies that any arrow can be resolved in an independent way:



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In our case, we also make a few more technical requirements, e.g. closure under concrete directed colimits, independent squares are pullbacks, and a strenghtening of accessibility, close to requiring "finitely accessible". We call the result a *superstable independence notion*. From now on, assume we are working in an AEC with a superstable independence notion.

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Theorem (Boney-Grossberg; V.)

Let $\mathbf{K}$ be an AEC, let $\kappa > \text{LS}(\mathbf{K})$ be strongly compact. If $\mathbf{K}$ is categorical in some $\mu > \kappa^{+\omega}$, then there is a superstable independence notion on the class $\mathbf{K}_{\geq \mu}$, and more generally on the subclass of $\kappa^{+\omega}$-model-homogeneous models of $\mathbf{K}$.

An I-system $\langle M_u : u \in I \rangle$ is called *independent* if whenever $u, v \subseteq w$ are in I, the square induced by $M_{u \cap v}, M_u, M_v, M_w$ is independent.

From now on, we say "system" instead of "independent system". We strenghten the (λ, n)-extension property in the expected way.

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The superstable independence notion gives $(\lambda, \leq 2)$-extension and $(\lambda, \leq 2)$-uniqueness for all λ.

To make it easier to describe results, define:

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K *has the* (λ, n)*-properties* if it has both (λ, n)-uniqueness and (λ, n)-extension.

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If $\mathbf{K}$ has the $(< \lambda, \leq n)$-properties, then $\mathbf{K}$ has the $(\lambda, < n)$-properties.

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Corollary (Shelah, 1983)

If **K** has the $(\aleph_0, < \omega)$-properties, then **K** has the $(\lambda, < \omega)$-properties for every λ.

From now on, drop the assumption that $\mathrm{LS}(\mathbf{K}) = \aleph_0$.

Definition

Call $\mathbf{K}$ *excellent* if it has the $(\mathrm{LS}(\mathbf{K}), < \omega)$-properties.

We have just seen that excellent AECs have in fact the $(\lambda, < \omega)$-properties for all λ. After a little bit more work, one gets a categoricity transfer:

Theorem (Shelah-V.)

The categoricity spectrum of an excellent AEC $\mathbf{K}$ is either empty, $\{\mathrm{LS}(\mathbf{K})\}$, or an end-segment starting below $\beth_{\left(2^{\mathrm{LS}(\mathbf{K})}\right)^+}$.

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Great... But how does one get excellence???

Limit systems

Definition

For M, N of the same cardinality, N is *limit over M* if there exists an increasing continuous chain $\langle M_i : i \leq \delta \rangle$ such that $M_0 = M$, $M_\delta = M$, and M_{i+1} is universal over M_i for all $i < \delta$.

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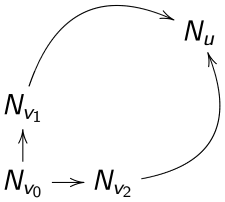

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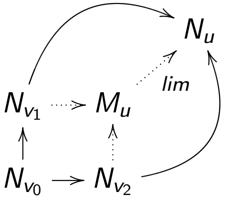

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If **K** has a superstable independence notion, then **K** has strong $(\lambda, \leq 1)$-uniqueness for every λ.

Also define limit (λ, n)-extension (limit (λ, n^-)-systems can be extended to limit (λ, n)-systems), limit (λ, n)-existence (limit (λ, n)-system exists), and limit (λ, n)-uniqueness (like (λ, n)-uniqueness but for limit models).

Lemma

1. Limit $(\lambda, \leq n)$-extension implies limit $(\lambda, \leq n)$-existence.
2. Limit $(\lambda, \leq 2)$-extension and strong $(\lambda, \leq 2)$-uniqueness hold.
3. $(\lambda, \leq n)$-extension implies limit $(\lambda, \leq n)$-extension.
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Assume categoricity in λ.

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Bottom line: it suffices to look at the limit properties.

Lemma

Limit $(\lambda, < n)$-uniqueness implies that any two limit (λ, n^{-})-systems are isomorphic.

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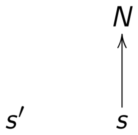

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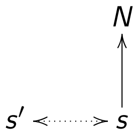

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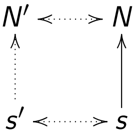


Getting existence

Lemma

Limit (λ^+, n)-existence implies limit $(\lambda, n+1)$-existence.

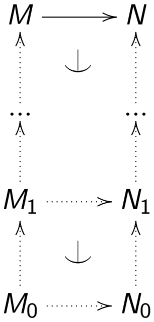
Proof.

Resolve the system! □

Corollary

Limit $([\lambda, \lambda^{+\omega}), 0)$-existence implies limit $([\lambda, \lambda^{+\omega}), < \omega)$-existence.

$$M \longrightarrow N$$


M
N
M1
N1
M0
N0

Getting uniqueness

We have seen how to derive existence and extension. It remains to deal with uniqueness. The template is:

Fact (Shelah, 1987)

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The basic idea is to proceed by contradiction, build a continuous tree of failure, then use the cardinal arithmetic (weak diamond) to get a contradiction.

Getting uniqueness without cardinal arithmetic

Theorem (Shelah-V.)

Let θ be least such that $2^\lambda < 2^\theta$. If **K** has the limit $([\lambda, \theta], \leq n)$-properties, then there *exists* unboundedly-many $\lambda' \in [\lambda, \theta)$ such that **K** has $(\lambda', n+1)$-uniqueness.

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If we work above some large cardinals (e.g. a strongly compact cardinal), then sufficiently complete ultraproduct of systems are well-behaved, so (λ', n)-uniqueness transfers down, and we get $([\lambda, \theta), n+1)$-uniqueness.

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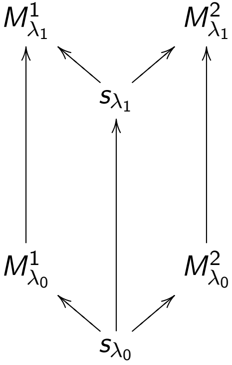
Theorem (Shelah-V.)

Let θ be least such that $2^\lambda < 2^\theta$. Assume that systems can be "uniformly extended". If **K** has the limit $([\lambda, \theta], \leq n)$-properties, then **K** has limit $([\lambda, \theta), n+1)$-uniqueness.

$$\begin{array}{ccc} M^1_{\lambda_0} & & M^2_{\lambda_0} \\ & \nwarrow \quad \nearrow & \\ & s_{\lambda_0} & \end{array}$$



$M^1_{\lambda_1}$
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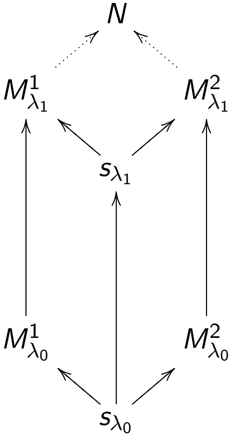
N
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Putting it all together

Recall that we are working in an AEC $\mathbf{K}$ with a superstable independence notion. This implies that all the 2-dimensional properties hold. Therefore we obtain:

Theorem (Shelah-V.)

Let $\mathbf{K}$ be an AEC with a superstable independence notion. Assume that $\mathbf{K}$ is categorical in $\lambda := \text{LS}(\mathbf{K})$.

1. If $2^{\lambda^{+n}} < 2^{\lambda^{+(n+1)}}$ for all $n < \omega$, then $\mathbf{K}$ is excellent.
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The categoricity assumption is not too important: one can show that if $\mathbf{K}$ is an AEC with a superstable independence notion, then for any $\mu > \mathrm{LS}(\mathbf{K})$, the class of μ-saturated models is an AEC with a superstable independence notion that is categorical in μ.

Some remarks

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Theorem (V.)

Let ψ be a universal $\mathbb{L}_{\omega_1,\omega}$ sentence. If ψ is categorical in *some* $\mu \geq \beth_{\beth_{\omega_1}}$, then ψ is categorical in *all* $\mu' \geq \beth_{\beth_{\omega_1}}$.

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This has recently been extended to the more general *multiuniversal classes* (Ackerman-Boney-V.).

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- Is multidimensional independence canonical?

Canonicity of multidimensional independence

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Are there other possible notions of multidimensional independence, or do two dimensions suffice?

Thank you!

Some references:

- Saharon Shelah and Sebastien Vasey, *Categoricity and multidimensional diagrams*, arXiv:1805.0629.
- Michael Lieberman, Jiří Rosický, and Sebastien Vasey, *Forking independence from the categorical point of view*, arXiv:1801.09001.
- Sebastien Vasey, *The categoricity spectrum of large abstract elementary classes with amalgamation*, arXiv:1805.04068.
- Rami Grossberg, *Classification theory for abstract elementary classes*. Logic and Algebra, ed. Yi Zhang, Contemporary Mathematics, Vol 302, AMS, (2002), 165–204.
- Saharon Shelah, *Classification theory for abstract elementary classes*. Studies in Logic: Mathematical logic and foundations, vol. 18 & 20, College Publications. 2009 [Introduction available online: E53 on Shelah's list].
- John T. Baldwin, *Categoricity*. University Lecture Series, vol. 50, American Mathematical Society, 2009.