Non-elementary classification theory

Sebastien Vasey

Harvard University

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Example

- Sets (with no structure) are categorical in all infinite cardinals.
- $\mathbb{Q}$-vector spaces and algebraically closed fields of characteristic zero are categorical exactly in the uncountable cardinals.
- Dense linear orders without endpoints are categorical only in $\aleph_0$.

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Question

What if K is not first-order axiomatizable? For example, what if K is axiomatized by an infinitary logic?

Shelah's eventual categoricity conjecture ($\sim$1977)

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There is a cardinal μ such that a *universal* $\mathbb{L}_{\omega_1,\omega}$ sentence categorical in *some* $\lambda \geq \mu$ is categorical in *all* $\lambda' \geq \mu$.

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Note: Shelah conjectured $\mu = \beth_{\omega_1}$ (the lowest it can be). In the theorem, $\mu = \beth_{\beth_{\omega_1}}$. The spirit is that we look "high-enough" as low cardinals are more prone to pathologies/coding tricks (c.f. the behavior of DLOs). In many earlier approximations, μ was a large cardinal.

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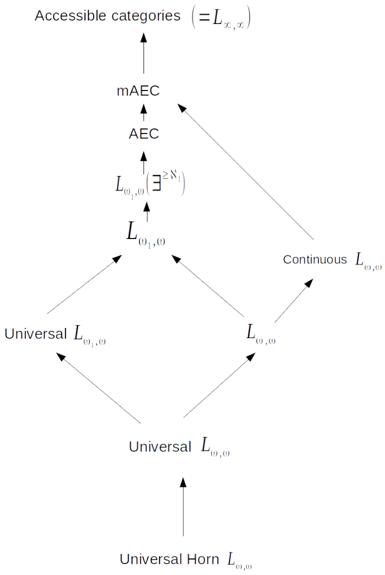
5. A sketch of the proof of SECC in universal classes.
6. Conclusion.

Frameworks for model theory



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Algebraically closed fields are *not* a universal class.

Fact (Tarski, 1954)

K is universal if and only if K is the class of models of a universal $\mathbb{L}_{\infty,\omega}$-theory.

Eventual categoricity in universal classes

Theorem (V., 2017)

Let K be a universal class. There is a cardinal μ (depending only on $|\tau(K)|$) such that if K is categorical in *some* $\lambda \geq \mu$, then K is categorical in *all* $\lambda' \geq \mu$.

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Note: in fact one can take $\mu = \beth_{\beth_{\left(2^{|\tau(K)|}\right)^+}}$.

Abstract elementary classes (Shelah, 1980s)

An AEC is a pair $\mathbf{K} = (K, \leq_{\mathbf{K}})$, where K is a class of structures in a fixed vocabulary $\tau(\mathbf{K})$ and $\leq_{\mathbf{K}}$ is a partial order on $\mathbf{K}$ satisfying some of the basic category-theoretic properties of $(\mathrm{Mod}(T), \preceq)$.

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There exists a (least) cardinal $\mathrm{LS}(\mathbf{K}) \geq |\tau(\mathbf{K})| + \aleph_0$ such that for any $M \in \mathbf{K}$ and any $A \subseteq |M|$, there is $M_0 \leq_{\mathbf{K}} M$ containing A with $\|M_0\| \leq |A| + \mathrm{LS}(\mathbf{K})$.

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Examples include $(\mathrm{Mod}(T), \preceq)$ (where $\mathrm{LS}(\mathbf{K}) = |T|$), $\mathbf{K} = (K, \subseteq)$ where K is a universal class ($\mathrm{LS}(\mathbf{K}) = |\tau(\mathbf{K})| + \aleph_0$), $(\mathrm{Mod}(\psi), \preceq_{\Phi})$ (where $\mathrm{LS}(\mathbf{K}) = |\Phi| + |\tau(\Phi)| + \aleph_0$), $\psi \in \mathbb{L}_{\infty,\omega}$, and more generally classes of models of $\mathbb{L}_{\infty,\omega}(\exists^{\geq\lambda})$ sentences.

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Even if the class is elementary, the ordering may not be elementary substructure. E.g. fields ordered by subfield or classes K of modules with $M \leq_{\mathbf{K}} N$ iff $N/M \in K$ (Baldwin-Eklof-Trlifaj, 2007).

Shelah's eventual categoricity conjecture for AECs

An AEC categorical in *some* high-enough cardinal is categorical in *all* high-enough cardinals.

Accessible categories

Definition (Gabriel-Ulmer, 1971)

An object M in a category $\mathcal{C}$ is *λ-presentable* (λ regular) if any morphism of M into the colimit of a λ-directed system factors through the system.

Definition (Lair 1981, Makkai-Paré 1989)

A category $\mathcal{C}$ is *λ-accessible* if it is closed under λ-directed colimits, contains a set of λ-presentable objects, and every object is a λ-directed colimit of λ-presentable objects. $\mathcal{C}$ is *accessible* if it is λ-accessible for some λ.

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Any AEC $\mathbf{K}$ (with morphisms the injective homomorphisms $f : M \to N$ such that $f[M] \leq_{\mathbf{K}} N$) is accessible, but there are other examples (Banach spaces, λ-complete Boolean algebras, etc.).

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Fact (Lieberman-Rosický-V., 2017 (preprint))

Universal classes are, up to equivalence of categories, locally $\aleph_0$-multipresentable categories whose morphisms are mono.

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The lack of compactness.

- An arbitrary AEC may fail amalgamation.
- Even if the AEC has amalgamation, Morley's proof does not generalize: what is the type of an element in this context?

Orbital types

In any AEC $\mathbf{K}$, we define $\text{tp}_{\mathbf{K}}(a/B; M)$ (the orbital type of a over B as computed inside M), for $M \in \mathbf{K}$, $a \in M$, and $B \subseteq |M|$, as the finest notion of type preserving $\mathbf{K}$-embeddings.

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More precisely, $\text{tp}_{\mathbf{K}}(a/B; M)$ is the E-equivalence class of (a, B, M), where E is the finest equivalence relation on pairs (a, B, M) satisfying:

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Example

Let $\mathbf{K}$ be the AEC consisting of isomorphic copies of $(\mathbb{Q}, <)$, with $M \leq_{\mathbf{K}} N$ iff $M = N$. Then $\text{tp}_{\mathbf{K}}(1/(0,1); \mathbb{Q}) \neq \text{tp}_{\mathbf{K}}(2/(0,1); \mathbb{Q})$, but both have the same syntactic type.

Tameness

Definition (Grossberg-Vandieren, 2006)

An AEC $\mathbf{K}$ is $(< \kappa)$-*tame* if whenever p, q are distinct orbital types over M, there exists $A \subseteq |M|$ with $|A| < \kappa$ such that $p \upharpoonright A \neq q \upharpoonright A$. We say that $\mathbf{K}$ is *tame* if it is $(< \kappa)$-tame for some κ.

The previous example was not $(< \aleph_0)$-tame.

Why is Shelah's eventual categoricity conjecture interesting?

Conjecture (Shelah)

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- Q: This is only a test question: what is the real goal?
- A: To develop a theory of independence, dimensions, and the dividing lines around it, in the very general setup of AECs.

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- AECs are very closed: if $\mathbf{K}$ is an AEC, then so is:
  1. $\mathbf{K}_{\geq \lambda}$, its class of models of cardinality at least λ.
  2. $\mathbf{K}_{\neg p}$, its class of models omitting a fixed type p.
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  These closure properties are used throughout the development of the theory. They do *not* hold for elementary classes (or even say $\mathbb{L}_{\infty,\omega}$). Thus studying AECs can help us understand elementary classes better.
- We have less absoluteness, so more interesting connections with set theory!

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This fails already for $\mathbb{L}_{\omega_1,\omega}(\exists^{\geq \aleph_1})$. In fact:

Theorem (Shelah)

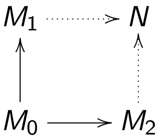
There is an $\mathbb{L}_{\omega_1,\omega}(\exists^{\geq \aleph_1})$-sentence ψ that is categorical in $\aleph_0$ such that:

1. If $2^{\aleph_0} > \aleph_1$ and $\mathsf{MA}_{\aleph_1}$ holds, then ψ is categorical in $\aleph_1$.
2. If $2^{\aleph_0} < 2^{\aleph_1}$, then ψ has $2^{\aleph_1}$-many models of cardinality $\aleph_1$.

Connections with combinatorial set theory

Theorem (Shelah, 1987)

Assume $2^\lambda < 2^{\lambda^+}$. Let $\mathbf{K}$ be an AEC which is categorical in λ and λ^+. Then $\mathbf{K}$ has the amalgamation property for models of cardinality λ.



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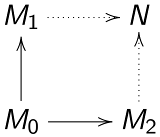

The proof proceeds by contradiction and uses the weak diamond to "guess" how to build a tree of failures and code many models in λ^+ using disjoint stationary sets.

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Corollary (Shelah, Grossberg-VanDieren, Boney, 2014)

Shelah's eventual categoricity conjecture *for a successor* holds for all AECs provided that there is a proper class of strongly compact cardinals.

Connections with large cardinals (2)

Theorem (V., 2016)

Assume $2^{\lambda} < 2^{\lambda^+}$ for all λ, there is a proper class of strongly compact cardinals, and a claim of Shelah holds. Then Shelah's eventual categoricity conjecture holds for all AECs.

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If there is a proper class of strongly compact cardinals, then Shelah's eventual categoricity conjecture holds for all AECs closed under intersections.

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Theorem (Boney-Unger, 2017)

There is an AEC which is tame *only if* a large cardinal axiom holds. Thus the statement "every AEC is tame" is *equivalent* to a large cardinal axiom (a proper class of almost strongly compact cardinals).

Compactness, large cardinals, and stability

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The equation is:

"Amount of model-theoretic compactness = Amount of set-theoretic compactness + amount of stability".

Large cardinals and stability: an example

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- "$(*)_T$ for all T" is a large cardinal axiom (implying the existence of $0^\sharp$).

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$(*)_T$ Every long-enough sequence in a model of T contains an indiscernible subsequence.

- "$(*)_T$ for all T" is a large cardinal axiom (implying the existence of $0^\sharp$).
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- Conclusion: one should attempt to isolate the model-theoretic properties provided by large cardinals (e.g. tameness), and study them separately, trying in particular to derive them from stability-theoretic assumptions (e.g. categoricity).
- **Conjecture** (Grossberg 1986, Grossberg-VanDieren 2006): Amalgamation and tameness should follow from categoricity.

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Theorem

Let $\mathbf{K}$ be a tame AEC with amalgamation.

- (Grossberg-V. 2017) In tame AECs with amalgamation, several of the usual definitions of superstability are equivalent.
- (V. 2017 (preprint)) In tame AECs with amalgamation, superstability follows from stability on a tail.

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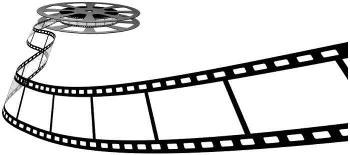
- Upside: very general and powerful.
- Downside: complex, often not in ZFC.

Good frames (Shelah, Sh:600, 2009)



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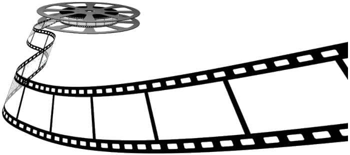

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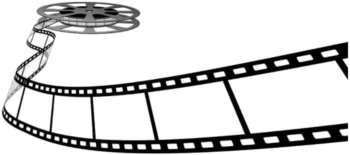

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Theorem (Shelah, 2009)

($2^\lambda < 2^{\lambda^+} < 2^{\lambda^{++}}$) If **K** has a good λ-frame, then either there are many models in λ^{++}, or a subclass of **K** has a good λ^+-frame.

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Theorem (V., 2017 (preprint))

$(2^\lambda < 2^{\lambda^+})$ If **K** has a (categorical) good λ-frame and a good λ^+-frame, then it is (λ, λ^+)-tame.

It follows that two consecutive good frames are "connected": this is a result deriving compactness from stability assumptions (and a little bit of combinatorial set theory, but no large cardinals).

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Let K be a universal class categorical in a high-enough cardinal. Then there is an ordering $\leq$ such that $(K, \leq)$ is a tame AEC with amalgamation and primes (over sets of the form Ma).

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The second step proves the categoricity transfer in the new class:

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Shelah's eventual categoricity conjecture holds in tame AECs with amalgamation and primes.

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Lemma

If K is a tame AECs with primes and amalgamation categorical in *two* "high-enough" cardinals $\lambda_1 < \lambda_2$, then K is categorical in λ_1^+.

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5. By tameness there is a good λ_2-frame on $\mathbf{K}_{\neg p}$, so the model in λ_2 omits p, contradiction!

□

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- Classification theory can be developped for non-elementary classes...
- ...but much remains to be done.
- For starters, Shelah's eventual categoricity conjecture for AECs is still not known to be consistent with large cardinals (although there are strong indications).
- Connections with set theory and category theory are worth exploring more.
- Strong interplay between model-theoretic compactness, stability, and set-theoretic compactness (e.g. diamond-like principles or large cardinals).

Thank you!

Some references:

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