Forking and categoricity in non-elementary model theory

Sebastien Vasey

Harvard University

August 8, 2019
Some recent directions in model theory
16th CLMPST
Prague

Categories of models

Given a first-order theory T, we want to study its class K of models as a category. What should the morphisms be?

Categories of models

Given a first-order theory T, we want to study its class K of models as a category. What should the morphisms be?

- ("West coast" model theory) Elementary embeddings.

Categories of models

Given a first-order theory T, we want to study its class K of models as a category. What should the morphisms be?

- ("West coast" model theory) Elementary embeddings.
- ("East coast" model theory) Monomorphisms, aka substructure embeddings.

Categories of models

Given a first-order theory T, we want to study its class K of models as a category. What should the morphisms be?

- ("West coast" model theory) Elementary embeddings.
- ("East coast" model theory) Monomorphisms, aka substructure embeddings.
- (Universal algebra) Homomorphisms.

Categories of models

Given a first-order theory T, we want to study its class K of models as a category. What should the morphisms be?

- ("West coast" model theory) Elementary embeddings.
- ("East coast" model theory) Monomorphisms, aka substructure embeddings.
- (Universal algebra) Homomorphisms.

Let $\mathrm{Elem}(T)$, $\mathrm{Emb}(T)$, and $\mathrm{Mod}(T)$ be the respective categories.

We will focus on $\mathrm{Elem}(T)$ at first, then will give a framework encompassing all of these and much more.

Category-theoretic properties of Elem(T)

Question

What traditional model-theoretic properties can we "see" by just looking at the category Elem(T)? That is, which of these properties are invariant under equivalence of categories?

Category-theoretic properties of Elem(T)

Question

What traditional model-theoretic properties can we "see" by just looking at the category Elem(T)? That is, which of these properties are invariant under equivalence of categories?

- Non-examples: quantifier elimination (Morleyization gives an isomorphic category). Elimination of imaginaries (passing to T^{eq} gives an equivalent category).

Category-theoretic properties of Elem(T)

Question

What traditional model-theoretic properties can we "see" by just looking at the category Elem(T)? That is, which of these properties are invariant under equivalence of categories?

- Non-examples: quantifier elimination (Morleyization gives an isomorphic category). Elimination of imaginaries (passing to T^{eq} gives an equivalent category).
- Examples: categoricity in λ, stability in λ (but for this we need to describe cardinalities category-theoretically!).

Presentability

Let λ be a regular cardinal.

Definition

A partially ordered set is λ-*directed* if any subset of size strictly less than λ has an upper bound. Directed means $\aleph_0$-directed.

Presentability

Let λ be a regular cardinal.

Definition

A partially ordered set is λ-*directed* if any subset of size strictly less than λ has an upper bound. Directed means $\aleph_0$-directed.

Definition (Gabriel-Ulmer, 1971)

An object M in a category $\mathcal{K}$ is λ-*presentable* if for any λ-directed diagram $D : I \to \mathcal{K}$, any morphism $f : M \to \operatorname{colim} D$ factors through some D_i:

$$\begin{array}{ccc} M & \longrightarrow & \operatorname{colim} D \\ & & \uparrow \\ & & D_i \end{array}$$

Presentability

Let λ be a regular cardinal.

Definition

A partially ordered set is λ-*directed* if any subset of size strictly less than λ has an upper bound. Directed means $\aleph_0$-directed.

Definition (Gabriel-Ulmer, 1971)

An object M in a category $\mathcal{K}$ is λ-*presentable* if for any λ-directed diagram $D : I \to \mathcal{K}$, any morphism $f : M \to \operatorname{colim} D$ factors through some D_i:

$$\begin{array}{ccc} M & \longrightarrow & \operatorname{colim} D \\ & \searrow & \uparrow \\ & & D_i \end{array}$$

Presentability: more definitions

Example

In the category of sets, an object is λ-presentable if and only if it has cardinality strictly less than λ.

Presentability: more definitions

Example

In the category of sets, an object is λ-presentable if and only if it has cardinality strictly less than λ.

Definition

The *presentability rank* of M is the minimal regular cardinal λ so that M is λ-presentable.

Presentability: more definitions

Example

In the category of sets, an object is λ-presentable if and only if it has cardinality strictly less than λ.

Definition

The *presentability rank* of M is the minimal regular cardinal λ so that M is λ-presentable.

Definition

The *(internal) size* of M is the predecessor of its presentability rank (if it exists).

Some category-theoretic properties

Work in $\mathrm{Elem}(T)$. Let $\lambda > |T|$.

Some category-theoretic properties

Work in Elem(T). Let $\lambda > |T|$.

- M has cardinality λ if and only if M has internal size λ.

Some category-theoretic properties

Work in Elem(T). Let $\lambda > |T|$.

- M has cardinality λ if and only if M has internal size λ.
- T is categorical in λ if and only if Elem(T) has a unique object of size λ, up to isomorphism.

Some category-theoretic properties

Work in $\mathrm{Elem}(T)$. Let $\lambda > |T|$.

- M has cardinality λ if and only if M has internal size λ.
- T is categorical in λ if and only if $\mathrm{Elem}(T)$ has a unique object of size λ, up to isomorphism.
- T is stable in λ if and only if for every object M of size λ, there exists N of size λ universal over M.

Some category-theoretic properties

Work in Elem(T). Let $\lambda > |T|$.

- M has cardinality λ if and only if M has internal size λ.
- T is categorical in λ if and only if Elem(T) has a unique object of size λ, up to isomorphism.
- T is stable in λ if and only if for every object M of size λ, there exists N of size λ universal over M.
- At least consistently, we can also characterize simplicity (using saturation) and NIP (using the generic pair theorem).

Some category-theoretic properties

Work in $\text{Elem}(T)$. Let $\lambda > |T|$.

- M has cardinality λ if and only if M has internal size λ.
- T is categorical in λ if and only if $\text{Elem}(T)$ has a unique object of size λ, up to isomorphism.
- T is stable in λ if and only if for every object M of size λ, there exists N of size λ universal over M.
- At least consistently, we can also characterize simplicity (using saturation) and NIP (using the generic pair theorem).

Shelah's thesis: dividing lines should have "internal" and "external" characterizations.

Definition (Lair 81, Makkai-Paré 89, Adámek-Rosický 94)

A category is λ-*accessible* if:

1. It has λ-directed colimits.
2. There is a set S of λ-presentable objects such that any object is a λ-directed colimits of members of S.

Definition (Lair 81, Makkai-Paré 89, Adámek-Rosický 94)

A category is λ-*accessible* if:

1. It has λ-directed colimits.
2. There is a set S of λ-presentable objects such that any object is a λ-directed colimits of members of S.

Example

- Elem(T), Emb(T), Mod(T), for T first-order (in the last two cases, of "low" complexity). This works even if T has sentences in $\mathbb{L}_{\infty,\infty}$.

Definition (Lair 81, Makkai-Paré 89, Adámek-Rosický 94)

A category is λ-*accessible* if:

1. It has λ-directed colimits.
2. There is a set S of λ-presentable objects such that any object is a λ-directed colimits of members of S.

Example

- Elem(T), Emb(T), Mod(T), for T first-order (in the last two cases, of "low" complexity). This works even if T has sentences in $\mathbb{L}_{\infty,\infty}$.
- R-Mod, with homomorphisms or monomorphisms or pure monomorphisms, etc.

Definition (Lair 81, Makkai-Paré 89, Adámek-Rosický 94)

A category is λ-*accessible* if:

1. It has λ-directed colimits.
2. There is a set S of λ-presentable objects such that any object is a λ-directed colimits of members of S.

Example

- Elem(T), Emb(T), Mod(T), for T first-order (in the last two cases, of "low" complexity). This works even if T has sentences in $\mathbb{L}_{\infty,\infty}$.
- R-Mod, with homomorphisms or monomorphisms or pure monomorphisms, etc.
- Banach spaces with contractions (internal size is density character).

Definition (Lair 81, Makkai-Paré 89, Adámek-Rosický 94)

A category is λ-*accessible* if:

1. It has λ-directed colimits.
2. There is a set S of λ-presentable objects such that any object is a λ-directed colimits of members of S.

Example

- Elem(T), Emb(T), Mod(T), for T first-order (in the last two cases, of "low" complexity). This works even if T has sentences in $\mathbb{L}_{\infty,\infty}$.
- R-Mod, with homomorphisms or monomorphisms or pure monomorphisms, etc.
- Banach spaces with contractions (internal size is density character).

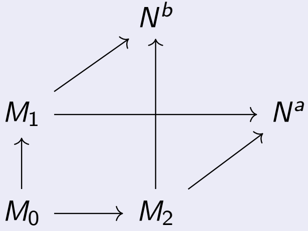
Definition (Equivalence of amalgam)

Consider a span: $M_1 \leftarrow M_0 \rightarrow M_2$

Definition (Equivalence of amalgam)

Consider a span: $M_1 \leftarrow M_0 \rightarrow M_2$

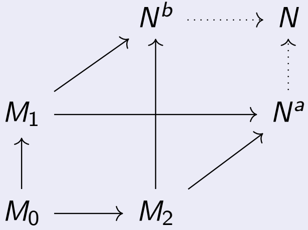
Two amalgams $M_1 \rightarrow N^a \leftarrow M_2$, $M_1 \rightarrow N^b \leftarrow M_2$ of this span are *equivalent* if there exists N and arrows making the following diagram commute:



Definition (Equivalence of amalgam)

Consider a span: $M_1 \leftarrow M_0 \rightarrow M_2$

Two amalgams $M_1 \rightarrow N^a \leftarrow M_2$, $M_1 \rightarrow N^b \leftarrow M_2$ of this span are *equivalent* if there exists N and arrows making the following diagram commute:



Example: in Set_{mono}, $\{0\}$ and $\{1\}$ have two non-equivalent amalgams over $\emptyset$: $\{0, 1\}$ and $\{1\}$ (with the obvious morphisms).

Definition (Lieberman-Rosický-V., 2019)

A *stable independence notion* is a class of squares (called *independent squares*, marked with $\downarrow$) such that:

Definition (Lieberman-Rosický-V., 2019)

A *stable independence notion* is a class of squares (called *independent squares*, marked with $\mathrel{\smile\mkern-12mu\mid}$) such that:

1. Independent squares are closed under equivalence of amalgam.

Definition (Lieberman-Rosický-V., 2019)

A *stable independence notion* is a class of squares (called *independent squares*, marked with $\downarrow$) such that:

1. Independent squares are closed under equivalence of amalgam.
2. Existence: any span can be amalgamated to an independent square.

Definition (Lieberman-Rosický-V., 2019)

A *stable independence notion* is a class of squares (called *independent squares*, marked with $\downarrow$) such that:

1. Independent squares are closed under equivalence of amalgam.
2. Existence: any span can be amalgamated to an independent square.
3. Uniqueness: any two *independent* amalgam of the same span are equivalent.

Definition (Lieberman-Rosický-V., 2019)

A *stable independence notion* is a class of squares (called *independent squares*, marked with $\downarrow$) such that:

1. Independent squares are closed under equivalence of amalgam.
2. Existence: any span can be amalgamated to an independent square.
3. Uniqueness: any two *independent* amalgam of the same span are equivalent.
4. Symmetry:

$$\begin{array}{ccc} M_1 & \longrightarrow & M_3 \\ \uparrow & \downarrow & \uparrow \\ M_0 & \longrightarrow & M_2 \end{array} \quad \Rightarrow \quad \begin{array}{ccc} M_2 & \longrightarrow & M_3 \\ \uparrow & \downarrow & \uparrow \\ M_0 & \longrightarrow & M_1 \end{array}$$

Definition (stable independence notion - continued)

5. Transitivity:

$$\begin{array}{ccccc} M_1 & \longrightarrow & M_3 & \longrightarrow & M_5 \\ \uparrow & \mathop{\smile}\limits^{|} & \uparrow & \mathop{\smile}\limits^{|} & \uparrow \\ M_0 & \longrightarrow & M_2 & \longrightarrow & M_4 \end{array} \quad \Rightarrow \quad \begin{array}{ccccc} M_1 & \longrightarrow & & \longrightarrow & M_5 \\ \uparrow & & \mathop{\smile}\limits^{|} & & \uparrow \\ M_0 & \longrightarrow & & \longrightarrow & M_4 \end{array}$$

Definition (stable independence notion - continued)

5. Transitivity:

$$\begin{array}{ccccc} M_1 & \longrightarrow & M_3 & \longrightarrow & M_5 \\ \uparrow & \perp & \uparrow & \perp & \uparrow \\ M_0 & \longrightarrow & M_2 & \longrightarrow & M_4 \end{array} \quad \Rightarrow \quad \begin{array}{ccccc} M_1 & \longrightarrow & & \longrightarrow & M_5 \\ \uparrow & & \perp & & \uparrow \\ M_0 & \longrightarrow & & \longrightarrow & M_4 \end{array}$$

6. Accessibility: the category whose objects are arrows and whose morphisms are independent squares is accessible. This implies that any arrow can be "resolved" in an independent way:

$$\begin{array}{ccc} M & \longrightarrow & N \\ \uparrow & \perp & \uparrow \\ M_i & \dashrightarrow & N_i \end{array}$$

Theorem (LRV 2019)

A category with directed colimits has *at most one* stable independence notion.

Theorem (LRV 2019)

A category with directed colimits has *at most one* stable independence notion.

Example

- In any accessible category with pushouts, the class of all squares form a stable independence notion.

Theorem (LRV 2019)

A category with directed colimits has *at most one* stable independence notion.

Example

- In any accessible category with pushouts, the class of all squares form a stable independence notion.
- A first-order T is stable if and only if $\mathrm{Elem}(T)$ has a stable independence notion (given by nonforking squares).

Theorem (LRV 2019)

A category with directed colimits has *at most one* stable independence notion.

Example

- In any accessible category with pushouts, the class of all squares form a stable independence notion.
- A first-order T is stable if and only if $\mathrm{Elem}(T)$ has a stable independence notion (given by nonforking squares).
- (LRV 2019) Assuming a large cardinal axiom, an accessible category with directed colimits and all morphisms monos has stable independence (on a cofinal full subcategory) if and only if it is stable in a proper class of cardinals.

Stable independence and cofibrant generation

Theorem (LRV)

Let $\mathcal{K}$ be an accessible, bicomplete category. Let $\mathcal{M}$ be a class of morphisms of $\mathcal{K}$ such that:

1. $\mathcal{M}$ contains all isomorphisms, is closed under transfinite compositions, pushouts, and retracts.
2. The induced category $\mathcal{K}_{\mathcal{M}}$ is accessible and closed under directed colimits in $\mathcal{K}$.
3. $\mathcal{M}$ is coherent: if $M_0 \xrightarrow{f} M_1 \xrightarrow{g} M_2$, $g, gf \in \mathcal{M}$, then $f \in \mathcal{M}$.

Stable independence and cofibrant generation

Theorem (LRV)

Let $\mathcal{K}$ be an accessible, bicomplete category. Let $\mathcal{M}$ be a class of morphisms of $\mathcal{K}$ such that:

1. $\mathcal{M}$ contains all isomorphisms, is closed under transfinite compositions, pushouts, and retracts.
2. The induced category $\mathcal{K}_{\mathcal{M}}$ is accessible and closed under directed colimits in $\mathcal{K}$.
3. $\mathcal{M}$ is coherent: if $M_0 \xrightarrow{f} M_1 \xrightarrow{g} M_2$, $g, gf \in \mathcal{M}$, then $f \in \mathcal{M}$.

Then $\mathcal{K}_{\mathcal{M}}$ has stable independence if and only if $\mathcal{M}$ is cofibrantly generated (i.e. can be generated from a subset using transfinite compositions, pushouts, and retracts).

New examples of stable independence

Corollary (LRV)

1. The category of flat R-modules with flat morphisms (more generally, any AEC of "roots of Ext") has stable independence.
2. Any Grothendieck topos restricted to regular monos has stable independence.
3. Any Grothendieck abelian category restricted to regular monos has stable independence.
4. Any combinatorial model category where all objects are cofibrant and whose cofibrations are coherent (e.g. monos) has stable independence, when restricted to its cofibrations.

Higher dimensional stable independence

- Recall that part of the axioms of stable independence is that the category $\mathcal{K}_{ind}$ whose objects are $\mathcal{K}$-arrows and whose morphisms are independent squares is accessible.

Higher dimensional stable independence

- Recall that part of the axioms of stable independence is that the category $\mathcal{K}_{ind}$ whose objects are $\mathcal{K}$-arrows and whose morphisms are independent squares is accessible.
- It makes sense to ask: does $\mathcal{K}_{ind}$ have a stable independence notion?

Higher dimensional stable independence

- Recall that part of the axioms of stable independence is that the category $\mathcal{K}_{ind}$ whose objects are $\mathcal{K}$-arrows and whose morphisms are independent squares is accessible.
- It makes sense to ask: does $\mathcal{K}_{ind}$ have a stable independence notion?
- If yes, the independent squares in $\mathcal{K}_{ind}$ correspond to independent *cubes* in $\mathcal{K}$.

Higher dimensional stable independence

- Recall that part of the axioms of stable independence is that the category $\mathcal{K}_{ind}$ whose objects are $\mathcal{K}$-arrows and whose morphisms are independent squares is accessible.
- It makes sense to ask: does $\mathcal{K}_{ind}$ have a stable independence notion?
- If yes, the independent squares in $\mathcal{K}_{ind}$ correspond to independent *cubes* in $\mathcal{K}$.
- We can then ask whether $(\mathcal{K}_{ind})_{ind}$ has stable independence, etc.
- Continuing in this fashion, we can inductively define when $\mathcal{K}$ has n-dimensional stable independence, for $n \geq 2$.

Higher dimensional stable independence

- Recall that part of the axioms of stable independence is that the category $\mathcal{K}_{ind}$ whose objects are $\mathcal{K}$-arrows and whose morphisms are independent squares is accessible.
- It makes sense to ask: does $\mathcal{K}_{ind}$ have a stable independence notion?
- If yes, the independent squares in $\mathcal{K}_{ind}$ correspond to independent *cubes* in $\mathcal{K}$.
- We can then ask whether $(\mathcal{K}_{ind})_{ind}$ has stable independence, etc.
- Continuing in this fashion, we can inductively define when $\mathcal{K}$ has n-dimensional stable independence, for $n \geq 2$.
- These n-dimensional diagrams are known in model theory (Shelah's main gap, Zilber's pseudoexponential fields, etc.) but the definitions are arguably complicated and ad-hoc.

Theorem (eventual categoricity)

Let $\mathcal{K}$ be a finitely accessible category with all morphisms monos (or just an AEC). Assume either:

1. (Shelah-V.) There is a proper class of strongly compacts.
2. (V.) $\diamondsuit_S$ for all stationary sets S, and $\mathcal{K}$ has no maximal objects.
3. (V. 2017) $\mathcal{K}$ has "low complexity" (locally multipresentable/universal)

If $\mathcal{K}$ is categorical in *some* high-enough cardinal, then $\mathcal{K}$ is categorical in *all* high-enough cardinals.

Theorem (eventual categoricity)

Let $\mathcal{K}$ be a finitely accessible category with all morphisms monos (or just an AEC). Assume either:

1. (Shelah-V.) There is a proper class of strongly compacts.
2. (V.) $\diamondsuit_S$ for all stationary sets S, and $\mathcal{K}$ has no maximal objects.
3. (V. 2017) $\mathcal{K}$ has "low complexity" (locally multipresentable/universal)

If $\mathcal{K}$ is categorical in *some* high-enough cardinal, then $\mathcal{K}$ is categorical in *all* high-enough cardinals.

The proof of the first two proceed by first deriving existence of n-dimensional stable independence at all cardinals λ, via an elaborate induction on n and λ [schematically, (n, λ^+)-stable implies $(n+1, \lambda)$-stable].

Theorem (eventual categoricity)

Let $\mathcal{K}$ be a finitely accessible category with all morphisms monos (or just an AEC). Assume either:

1. (Shelah-V.) There is a proper class of strongly compacts.
2. (V.) $\diamondsuit_S$ for all stationary sets S, and $\mathcal{K}$ has no maximal objects.
3. (V. 2017) $\mathcal{K}$ has "low complexity" (locally multipresentable/universal)

If $\mathcal{K}$ is categorical in *some* high-enough cardinal, then $\mathcal{K}$ is categorical in *all* high-enough cardinals.

The proof of the first two proceed by first deriving existence of n-dimensional stable independence at all cardinals λ, via an elaborate induction on n and λ [schematically, (n, λ^+)-stable implies $(n+1, \lambda)$-stable]. In the third case, we can bypass this.

Open problems

- Can one prove eventual categoricity in ZFC, at least for finitely accessible categories or AECs?
- What is the status of eventual categoricity for accessible categories? Accessible categories with directed colimits?
- How do presentability ranks behave in general accessible categories? For example, can there be an object of limit presentability rank? The answer is no under SCH (LRV, 2019)
- When does existence of two-dimensional independence imply existence of higher dimensional independence?
- What are more examples of stable independence in mainstream mathematics (and computer science)?

Thank you!

Some references:

- Sebastien Vasey, *Accessible categories, set theory, and model theory: an invitation*, arXiv:1904.11307.
- Michael Lieberman, Jiří Rosický, and Sebastien Vasey, *Forking independence from the categorical point of view*, Advances in Mathematics **346** (2019), 719–772.
- Michael Lieberman, Jiří Rosický, and Sebastien Vasey, *Weak factorization systems and stable independence*, arXiv:1904.05691.
- Saharon Shelah and Sebastien Vasey, *Categoricity and multidimensional diagrams*, arXiv:1805.0629.
- Sebastien Vasey, *The categoricity spectrum of large abstract elementary classes with amalgamation*, arXiv:1805.04068.
- Sebastien Vasey, *Shelah's eventual categoricity conjecture in universal classes: part II*, Selecta Mathematica **23** (2017), no. 2, 1469–1506.