Forking and categoricity in non-elementary model theory

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Motivation: the set-theoretic point of view

Compactness/reflexion in set theory:

- Large cardinals.
- GCH, SCH.
- Jensen's diamond.
- Singular compactness (almost free implies free, Silver's theorem, etc.).
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- Chang's conjecture.
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- Stability theory!

Stability implies "tame" infinite combinatorics

Theorem (Shelah)

If a first-order theory T is stable in λ, then any sequence of length λ^+ contains an indiscernible subsequence of length λ^+.

Thus the partition relation $\lambda^+ \to (\lambda^+)^{<\omega}_\lambda$ holds, *when relativized to the models of* T! (a similar statement holds in AECs)
Goal: study the relationship between set-theoretic and model-theoretic compactness. Do it in a general framework where stability theory can be developped.

Universal classes and AECs

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They are exactly the classes axiomatizable by a universal $\mathbb{L}_{\infty,\omega}$ sentence (Tarksi, 1954) and, up to equivalence of categories, locally multipresentable categories with all morphisms monos (Diers 1980, Lieberman-Rosický-V. 2019).

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An *abstract elementary class* (AEC) is a class of structures $\mathbf{K}$ with a partial order $\leq_{\mathbf{K}}$ satisfying some properties, including closure under unions of $\leq_{\mathbf{K}}$-chains and a downward LST axiom. The expected notion of $\mathbf{K}$-embedding makes any AEC into a category.

Any AEC is an accessible categories with concrete directed colimits and all morphisms concrete monos (Lieberman 2011, Beke-Rosický 2012, ...).

Types in AECs

If an AEC has amalgamation and joint embedding, it has a model-homogeneous and universal "monster model" $\mathfrak{C}$. Work inside $\mathfrak{C}$.

The *type* of an element b over a set A, written $\mathbf{tp}(b/A)$, is defined to be the orbit of b under the automorphisms of $\mathfrak{C}$ fixing A pointwise. $\mathbf{K}$ is *stable in* λ if it has λ-many types over every set of size λ. *Superstable* means stable on a tail.

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An AEC is $(< \chi)$-*tame* if two distinct types over a model are separated by a subset of size strictly less than χ. *Tame* means $(< \chi)$-tame for some χ. *Finitely tame* means $(< \aleph_0)$-tame. (*this is a weak replacement for compactness*)

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Unless there are large cardinals, AECs are not always tame. Near example: the AEC with only $(\mathbb{Q}, <)$, $A = (0, 1)$, $b = 1$, $c = 2$. Nontrivial examples: Baldwin-Shelah 2008, Baldwin-Kolesnikov 2009, Boney-Unger 2017.

Examples of AECs

- $(\mathrm{Mod}(T), \preceq_\Phi)$, T a theory in $\mathbb{L}_{\infty,\omega}$, Φ a fragment containing ϕ.
- $(\mathrm{Mod}(T), \subseteq)$, for T first-order $\forall\exists$. Finitely tame if it has amalgamation.
- R-Mod, ordered with substructure. Universal class, stable. Superstable if and only if R is left Noetherian (Eklof 1971, Mazari-Armida).
- R-Mod, ordered with pure substructure. Finitely tame, stable (Kucera and Mazari-Armida). Superstable if and only if R is left pure semisimple (Mazari-Armida).
- Flat R-modules, with flat embeddings ($M \leq_{\mathbf{K}} N$ iff N/M is flat). More generally AECs of "roots of Ext" (Baldwin-Eklof-Trlifaj 2007). Tame and stable (Lieberman-Rosický-V.).
- Zilber's quasiminimal classes. Up to isomorphism of concrete categories, they are the AECs with countable LST number, a prime model, intersections, and a unique generic type over every countable model (V. 2018).

More examples of AECs

- Algebraically closed rank one valued fields. Finitely tame, stable.
- Existentially closed difference fields with n commuting automorphisms (Hyttinen-Kangas). Finitely tame, supersimple.
- AECs of geometric lattices (Hyttinen-Paolini 2018).
- If **K** is an AEC, so is its restriction to cardinalities above λ, its class of models omitting a fixed type, or its class of λ-saturated models (if **K** is suitably superstable). *Elementary classes are not closed under any of these operations.*
- Hart-Shelah example (1990): for each fixed $n < \omega$, an AEC with LST number $\aleph_0$ categorical in $\aleph_0, \aleph_1, \ldots, \aleph_n$.
- Morley's example (1965): for each fixed $\alpha < \left(2^{\aleph_0}\right)^+$, there is an AEC with LST number $\aleph_0$ categorical exactly in the cardinals $\lambda \geq \beth_\alpha$.
- ...
- More known and many more unknown examples.

Three basic results of Shelah

- (The presentation theorem, Sh:88) Any AEC is the (functorial) reduct of a universal class. *Idea: add "Skolem functions". If the AEC has arbitrarily large models, one deduces some leftover compactness (e.g. existence of Ehrenfeucht-Mostowski models).*

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- (Amalgamation from successive categoricity, Sh:88) Assume $2^{\lambda} < 2^{\lambda^+}$. If an AEC is categorical in λ and λ^+, then it has amalgamation for models of cardinality λ. *Idea: suppose not, build a tree of failures then use the weak diamond, more precisely the principle Θ_{λ^+} (Devlin-Shelah 1978).*

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- (Saturated = model-homogeneous, Sh:300) In an AEC with amalgamation, a model is λ-saturated if and only if it is λ-model-homogeneous. *Idea: partial embeddings don't behave well in general, so embed "point by point" but using* **K***-embeddings.*

Shelah's eventual categoricity conjecture

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Shelah's eventual categoricity conjecture (late 70s, still open): The categoricity spectrum of an AEC is either bounded or contains an end segment.

It is a test question to (in particular) develop stability and superstability theory for AECs.

Stability theory in tame AECs

In an AEC **K** with amalgamation, joint embedding, arbitrarily large models *that is tame*:

- (V. 2018) Assume SCH. If **K** is stable, there is χ so that for high-enough λ, **K** is stable in λ if and only if $\lambda = \lambda^{<\chi}$.
- (Boney-V. 2017, Grossberg-V. 2017, V. 2018) One can connect in the expected way (an abstract notion of) forking independence, (super)stability in terms of counting types, and the behavior of saturated models. For example, **K** is superstable if and only if for high-enough λ, unions of chains of λ-saturated models are λ-saturated.
- Categoricity implies superstability (Shelah-Villaveces 1999, V. 2016, Boney-Grossberg-VanDieren-V. 2017).
- If **K** has prime over sets of the form *Ma*, the eventual categoricity conjecture holds for **K** (Sh:394, Grossberg-VanDieren 2006, V. 2018).

Successive categoricity

- (Sh:88) An $\mathbb{L}_{\omega_1,\omega}$-sentence categorical in $\aleph_0$ and $\aleph_1$ must have a model of size $\aleph_2$. *(This is a very weak form of compactness from successive categoricity)*

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- (Shelah-V.) Assume WGCH. Categoricity of an AEC in ω-many successive cardinals implies categoricity everywhere above.
- (Mazari-Armida and V. 2018) Assume WGCH. A *universal class* (in a countable language) categorical in $\aleph_0$ and $\aleph_1$ is categorical everywhere.
- (V.) A universal class (say in a countable language) categorical on an end-segment below $\beth_\omega$ is categorical everywhere above $\beth_\omega$.

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Eventual categoricity

The categoricity spectrum of an AEC **K** is either bounded or contains an end segment provided that any of the following holds:

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- (Shelah-V.) There is a strongly compact cardinal above the LST number of **K**.
- (V.) Diamond holds at every stationary set and **K** has no maximal models.
- (V.) WGCH holds and **K** has amalgamation and arbitrarily large models. In fact, in this case (say if the LST number is countable) the categoricity spectrum is either empty, $[\aleph_m, \aleph_n]$ for $m \leq n < \omega$, or $[\chi, \infty)$ for $\chi < \beth_{(2^{\aleph_0})^+}$. There are examples of each type.

Proof ideas

The proofs of eventual categoricity proceed by building notions of independence, understanding superstability at a fixed cardinal ("good frames"), and (in the non-universal cases) developping a theory of higher-dimensional independence to move structural properties across cardinals. At the end of the proof, ideas from the "successive categoricity" results are used to find members of AECs of models omitting a type and contradict categoricity.

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Some open problems

1. Is the eventual categoricity conjecture true (in ZFC)?
2. Find more occurences of (higher-dimensional) stable independence "in the wild".
3. (Shelah-V., 2018) Does tameness follow from $\aleph_0$-stability? More precisely, let $\mathbf{K}$ be an (analytic) AEC that has amalgamation in $\aleph_0$, is categorical in $\aleph_0$, and is stable in $\aleph_0$. Is $\mathbf{K}$ finitely tame for types over countable models?

Some references

Recent introductory references:

- Sebastien Vasey, *Accessible categories, set theory, and model theory: an invitation*, arXiv:1904.11307.
- Will Boney and Sebastien Vasey, *A survey on tame abstract elementary classes*, Beyond First Order Model Theory (José Iovino ed.), CRC Press (2017), 353–427.
- Will Boney, *Classification theory for tame abstract elementary classes*. Lecture notes. Accessible from Will Boney's webpage.
- Sebastien Vasey, *Lecture notes on model theory for abstract elementary classes*. Accessible from my webpage.

Other introductory references include Rami Grossberg's survey (*Classification theory for abstract elementary classes*, 2002), John Baldwin's book (*Categoricity*, 2009), and of course Shelah's not so introductory two volume book (*Classification theory for abstract elementary classes*, 2009).