Stability theory for concrete categories

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A puzzle

If six students come to a party, then three of them all know each other, or three of them all do not know each other. More formally and generally:

Theorem (Ramsey, 1930)

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The notation is due to Erdős and Rado. It means: for any set X with at least n elements and any coloring $F : \binom{X}{2} \to \{0, 1\}$, there exists $H \subseteq X$ with $|H| = k$ so that $F \restriction \binom{H}{2}$ is constant (we call H a *homogeneous set* for F).

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If $k = 3$, $n = 6$ suffices. If $k = 5$, the optimal value of n is not known.

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The theorem does *not* rule out a party with uncountably-many students where all friends/strangers groups (= homogeneous sets) are countable.

The set theorist's dream

For any infinite cardinal λ, if λ students come to a party, then there is a group of λ-many that all know each other or a group of λ-many that all do not know each other. That is:

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This is wrong for most cardinals λ.

The Sierpiński coloring

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Proof.

Fix a well-ordering $\lhd$ of the reals. Set $F(\{x, y\}) = 1$ when $x < y$ iff $x \lhd y$, and $F(\{x, y\}) = 0$ otherwise (F is called the *Sierpiński coloring*). Assume for a contradiction H is an uncountable homogeneous set for F. Without loss of generality, for $x, y \in H$, $x < y$ if and only if $x \lhd y$. As $\lhd$ is a well-ordering, each $x \in H$ has an immediate successor x' in H. Find a rational r_x between x and x'. Then $x \to r_x$ is an injection of H (uncountable) into the rationals (countable), contradiction. □

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The Sierpiński coloring relies on a well-ordering of the reals. What if we consider only "definable/simple" colorings?

A counterexample with an infinite number of colors

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In the reals, a countable set allows one to distinguish uncountably-many points. There are however many structures where this is not the case.

The set theorist's dream in the complex numbers

Proposition

If $F : [\mathbb{C}]^2 \to \{0, 1\}$ is a coloring of the unordered pairs of complex numbers in two colors *such that* $F(\{f(x), f(y)\}) = F(\{x, y\})$ *for any field automorphism* f *of* $\mathbb{C}$, then F has a homogeneous set of cardinality $|\mathbb{C}|$.

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This proves $|\mathbb{C}| \to |\mathbb{C}|_2$ but "relativized to $\mathbb{C}$" (for colorings preserved by automorphisms).

Types

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Definition

Given a concrete category **K** with amalgamation and an object A of **K**, a *type over* A is just a pair $(x, A \xrightarrow{f} B)$, with $x \in B$. Two types $(x, A \xrightarrow{f} B)$, $(y, A \xrightarrow{g} C)$ are considered *the same* if there exists maps h_1, h_2 so that $h_1(x) = h_2(y)$ and the following diagram commutes:

$$\begin{array}{ccc} B & \overset{h_1}{\dashrightarrow} & D \\ {\scriptstyle f}\uparrow & & \uparrow{\scriptstyle h_2} \\ A & \overset{g}{\longrightarrow} & C \end{array}$$

Types in fields, linear orders, and graphs

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If we restrict to graphs with finite degrees, we obtain again at most $\max(|V(G)|, \aleph_0)$ types over G

Definition (Stability)

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- (Eklof 1971, Mazari-Armida) The category of R-modules with embeddings is always stable, and superstable if and only if R is left Noetherian.
- (Kucera and Mazari-Armida) The category of R-modules with pure embeddings is always stable, and superstable if and only if R is left pure-semisimple.

The set theorist's dream in stable AECs

Theorem (V.)

If **K** is an abstract elementary class with amalgamation and **K** is stable in λ, then:

$$\lambda^+ \xrightarrow{\mathbf{K}} \left(\lambda^+\right)_\lambda$$

Here λ^+ is the cardinal right after λ.

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Here λ^+ is the cardinal right after λ.

The partition notation means that given objects $A \to B$ in **K** with $|A| = \lambda$, $|B| = \lambda^+$, if F is a coloring of pairs from B in λ-many colors so that any two pairs with the same type over A have the same color, then we can find a homogeneous set for F of cardinality λ^+.

What an abstract elementary class (AEC) is will be explained in the next slide. All the examples given so far are AECs.

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Definition (Shelah, late 1970s)

An abstract elementary class (AEC) is a concrete category **K** satisfying the following conditions:

- All morphisms are concrete monomorphisms (injections).
- **K** has concrete directed colimits (also known as direct limits – basically closure under unions of increasing chains).
- (Smallness condition) Every object is a directed colimit of a fixed set of "small" subobjects.

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Any AEC is an accessible category: a category with all sufficiently directed colimits satisfying a certain smallness condition.

Abstract elementary classes and logic

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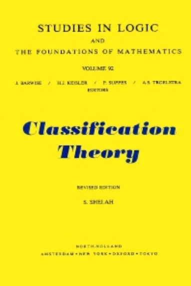
For any set T of first-order sentences, the category Mod(T) of models of T forms an AEC (the morphisms are the functions preserving all formulas).

We will call such a category a *first-order class*. It is one of the basic objects of study in model theory.

Stability theory was developped for first-order classes first, by Saharon Shelah.



Beyond first-order classes

There are some good reasons to look at more general classes. On the logic side, one can consider the infinitary logic $\mathbb{L}_{\infty,\omega}$, where infinite conjunctions and disjunctions are allowed (this logic also yields AECs, and usually any problem that is hard for AECs is hard already for this logic).

For example, we can say:

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First-order classes are important, because of the *compactness theorem*: if all finite subsets of a given theory have a model, then the whole theory has a model. This is powerful (one can use it to build models for nonstandard analysis) but means that many interesting categories are not first-order.

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Also, the *morphisms* of first-order classes are not so natural.

Examples

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- However the category of fields is not: while the axioms of fields are first-order, the embedding $\mathbb{Q} \to \mathbb{R}$ does not preserve all formulas (consider $(\exists x)(x \cdot x = 2)$).
- In fact none of the other examples given so far are first-order.

Eventual categoricity

Theorem (Morley, 1965)

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Morley's theorem was generalized to all first-order classes by Shelah (1974). He then asked about infinitary logics, and introduced AECs as a general framework to study the following question (*Shelah's eventual categoricity conjecture*).

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An AEC with a single object of *some* high-enough cardinality has a single object in *all* high-enough cardinalities.

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One motivating goal is to develop stability theory for AECs.

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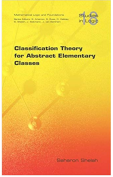
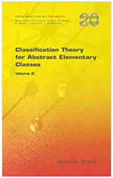
Partial approximations before my thesis include: Shelah 1983, Makkai-Shelah 1990, Shelah 1999, Shelah-Villaveces 1999, VanDieren 2006, Grossberg-VanDieren 2006, Shelah 2009, Hyttinen-Kesälä 2011, Boney 2014.





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Theorem (V. 2019)

Assuming the GCH, Shelah's eventual categoricity conjecture is true for AECs with amalgamation. In this case one can list all possibilities for the class of cardinals in which the category has a unique object.

Locality of types

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Not all AECs are tame in general. Boney (2014) showed tameness follows from a large cardinal axiom, and always holds in universal AECs.

Still, at the time there was some doubt about how reasonable it was to assume tameness.

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Assuming the GCH, any AEC with amalgamation and a unique object in some high-enough cardinal is tame.

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Theorem (Kucera and Mazari-Armida)

The AEC of R-modules with pure embeddings is tame.

Some characterizations of stability

Theorem (Stability spectrum, V. 2018)

Assume the GCH. For any stable tame AEC $\mathbf{K}$ with amalgamation, there is γ such that for all high-enough λ, $\mathbf{K}$ is stable in λ if and only if $\lambda = \lambda^{<\gamma}$.

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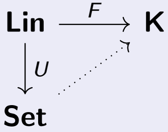
Theorem (V. 2016, Boney)

A tame AEC **K** with amalgamation is stable if and only if it does not have the "order property": any faithful functor $\mathbf{Lin} \xrightarrow{F} \mathbf{K}$ factors through the forgetful functor.



Order in graphs: an intermission

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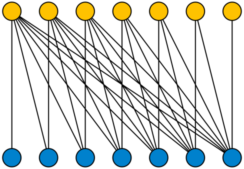
It is given by a half graph: for any linear ordering L, consider the bipartite graph on $L \sqcup L$ where we put an edge from i to j if only if $i \leq j$ (the picture below is for $L = \{1, 2, 3, 4, 5, 6, 7\}$):



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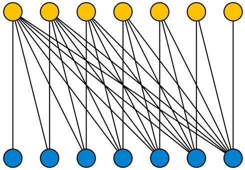

Graphs omitting half graphs are studied in finite combinatorics too (Malliaris-Shelah, *Regularity lemmas for stable graphs*. TAMS 2014).

Stable independence

The proofs of the eventual categoricity conjecture, of the stability spectrum theorem, and of the partition theorem $\lambda^+ \xrightarrow{\mathbf{K}} (\lambda^+)_\lambda$ involve describing what it means for a type to be "determined" over a small base. This is called forking in the first-order context, and is the key tool developped by Shelah in his classification theory book. It generalizes algebraic independence in fields.

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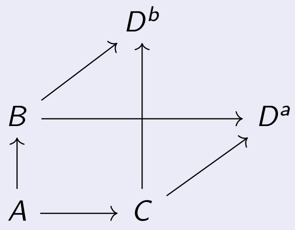
The proofs of the eventual categoricity conjecture, of the stability spectrum theorem, and of the partition theorem $\lambda^+ \xrightarrow{\mathbf{K}} (\lambda^+)_\lambda$ involve describing what it means for a type to be "determined" over a small base. This is called forking in the first-order context, and is the key tool developped by Shelah in his classification theory book. It generalizes algebraic independence in fields.

Unfortunately Shelah's definition is syntactic, hard to describe, and some properties depend on compactness. With my collaborators, we found a completely category-theoretic definition.

Definition (Equivalence of amalgam)

Consider a diagram: $B \leftarrow A \rightarrow C$

Two amalgams $B \rightarrow D^a \leftarrow C$, $B \rightarrow D^b \leftarrow C$ of this diagram are *equivalent* if there exists D and arrows making the following diagram commute:



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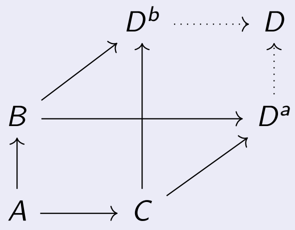

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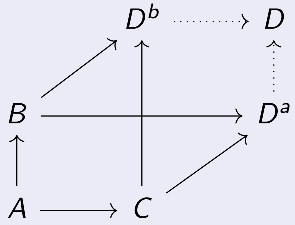

Example: in $\mathbf{Set}_{mono}$, $\{0\}$ and $\{1\}$ have two non-equivalent amalgams over $\emptyset$: $\{0, 1\}$ and $\{1\}$ (with the expected morphisms).

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A *stable independence notion* is a class of squares (called *independent squares*, marked with $\downarrow$) such that:

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A *stable independence notion* is a class of squares (called *independent squares*, marked with $\perp\!\!\!\smile$) such that:

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2. Existence: any span can be amalgamated to an independent square.
3. Uniqueness: any two *independent* amalgam of the same span are equivalent.
4. Symmetry:

$$\begin{array}{ccc} B & \longrightarrow & D \\ \uparrow & \perp\!\!\!\smile & \uparrow \\ A & \longrightarrow & C \end{array} \quad \Rightarrow \quad \begin{array}{ccc} C & \longrightarrow & D \\ \uparrow & \perp\!\!\!\smile & \uparrow \\ A & \longrightarrow & B \end{array}$$

Definition (stable independence notion - continued)

5. Transitivity:

$$\begin{array}{ccccc} B & \longrightarrow & D & \longrightarrow & F \\ \uparrow & \mathop{\smile}\limits^{\vert} & \uparrow & \mathop{\smile}\limits^{\vert} & \uparrow \\ A & \longrightarrow & C & \longrightarrow & E \end{array} \quad \Rightarrow \quad \begin{array}{ccccc} B & \longrightarrow & & \longrightarrow & F \\ \uparrow & & \mathop{\smile}\limits^{\vert} & & \uparrow \\ A & \longrightarrow & & \longrightarrow & E \end{array}$$

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A & \longrightarrow & & \longrightarrow & E
\end{array}
$$

6. Accessibility: the category whose objects are arrows and whose morphisms are independent squares is accessible. This implies that any arrow can be “filtered” in an independent way:

$$
\begin{array}{ccc}
M & \longrightarrow & N \\
\vdots\uparrow & \downfree & \vdots\uparrow \\
M_i & \cdots\cdots\rightarrow & N_i
\end{array}
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Theorem (Canonicity theorem; Lieberman-Rosický-V. 2019)

A category with directed colimits (in particular an AEC) has *at most one* stable independence notion.

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Theorem (Lieberman-Rosický-V. 2019)

An AEC with a stable independence notion has amalgamation, is tame, and is stable.

Certain converses are true too (for example in first-order classes, or assuming large cardinals).

Stable independence and cofibrant generation

Theorem (Lieberman-Rosický-V.)

Let $\mathcal{K}$ be an accessible, bicomplete category (like the category of R-modules with homomorphisms). Let $\mathcal{M}$ be a class of morphisms of $\mathcal{K}$ such that:

1. $\mathcal{M}$ contains all isomorphisms, is closed under transfinite compositions, pushouts, and retracts.
2. The induced category $\mathcal{K}_{\mathcal{M}}$ is accessible and closed under directed colimits in $\mathcal{K}$.
3. $\mathcal{M}$ is coherent: if $A \xrightarrow{f} B \xrightarrow{g} C$, $g, gf \in \mathcal{M}$, then $f \in \mathcal{M}$.

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Then $\mathcal{K}_{\mathcal{M}}$ has stable independence if and only if $\mathcal{M}$ is cofibrantly generated (i.e. can be generated from a subset using transfinite compositions, pushouts, and retracts).

New examples of stable independence

Corollary (Lieberman-Rosický-V.)

1. The AEC of flat R-modules with flat morphisms (more generally, any AEC of "roots of Ext") has stable independence.

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3. Any Grothendieck abelian category restricted to monos has stable independence.
4. Any Cisinski model category restricted to monos has stable independence.

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The category-theoretic definition of stable independence also naturally yields higher-dimensional generalizations (independent cubes). These are well known in model theory but the earlier definitions are ad-hoc and complicated. The goal is now to *develop a systematic theory, and also to find more examples.*

Thank you!

Some references:

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