Stability theory generalized to accessible categories

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A puzzle

If six students come to a party, then three of them all know each other, or three of them all do not know each other. More formally and generally:

Theorem (Ramsey, 1930)

For any natural number k, there exists a natural number n such that:

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The notation is due to Erdős and Rado. It means: for any set X with at least n elements and any coloring $F : \binom{X}{2} \to \{0, 1\}$, there exists $H \subseteq X$ with $|H| = k$ so that $F \upharpoonright \binom{H}{2}$ is constant (we call H a *homogeneous set* for F).

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If $k = 3$, $n = 6$ suffices. If $k = 5$, the optimal value of n is not known.

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The theorem does *not* rule out a party with uncountably-many students where all friends/strangers groups (= homogeneous sets) are countable.

The set theorist's dream

For any infinite cardinal λ, if λ students come to a party, then there is a group of λ-many that all know each other or a group of λ-many that all do not know each other. That is:

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This is wrong for most cardinals λ.

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Proof.

Fix a well-ordering $\lhd$ of the reals. Set $F(\{x, y\}) = 1$ when $x < y$ iff $x \lhd y$, and $F(\{x, y\}) = 0$ otherwise (F is called the *Sierpiński coloring*). Assume for a contradiction H is an uncountable homogeneous set for F. Without loss of generality, for $x, y \in H$, $x < y$ if and only if $x \lhd y$. As $\lhd$ is a well-ordering, each $x \in H$ has an immediate successor x' in H. Find a rational r_x between x and x'. Then $x \to r_x$ is an injection of H (uncountable) into the rationals (countable), contradiction. □

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The Sierpiński coloring relies on a well-ordering of the reals. What if we consider only "definable/simple" colorings?

A counterexample with an infinite number of colors

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In the reals, a countable set allows one to distinguish uncountably-many points. There are however many structures where this is not the case.

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Define inductively $(a_\alpha)_{\alpha<\lambda}$ by picking $a_\alpha \in X - \mathrm{acl}(\{a_i \mid i < \alpha\})$. This is always possible: the latter set has cardinality at most $\max(|\alpha|, \aleph_0) < \lambda$. ☐

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This proves $\lambda \to (\lambda)^{<\omega}_2$ "relativized to the complex numbers" (for colorings preserved by automorphisms).

Types

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Definition

Given a concrete category **K** with amalgamation and an object A of **K**, a *type over* A is just a pair $(x, A \xrightarrow{f} B)$, with $x \in B$. Two types $(x, A \xrightarrow{f} B)$, $(y, A \xrightarrow{g} C)$ are considered *the same* if there exists maps h_1, h_2 so that $h_1(x) = h_2(y)$ and the following diagram commutes:

$$\begin{tikzcd}
B \arrow[r, dotted, "h_1"] & D \\
A \arrow[u, "f"] \arrow[r, "g"] & C \arrow[u, dotted, "h_2"']
\end{tikzcd}$$

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If we restrict to graphs with finite degrees, we obtain again at most $\max(|V(G)|, \aleph_0)$ types over G

Definition (Stability)

A concrete category **K** is *stable in* λ if there are at most λ-many types over any object of cardinality λ. *Stable* means stable in an unbounded class, and *superstable* means stable on an end-segment. *Unstable* means not stable.

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- (Lieberman-Rosický-V.) The category of flat R-modules with flat monos is stable. True more generally for AECs of "Roots of Ext" (Baldwin-Eklof-Trlifaj 2007).

The set theorist's dream in stable AECs

Theorem (V.)

If **K** is an abstract elementary class with amalgamation and **K** is stable in λ, then:

$$\lambda^+ \xrightarrow{\mathbf{K}} (\lambda^+)^{<\omega}_{\lambda}$$

Here λ^+ is the cardinal right after λ.

The partition notation means that given objects $A \to B$ in **K** with $|A| \leq \lambda$, $|B| = \lambda^+$, and a λ^+-sized $X \subseteq B$, we can find $H \subseteq X$ also of size λ^+ in which any two finite subsequences of the same length have the same type over A.

What an abstract elementary class (AEC) is will be explained in the next slide. All the examples given so far are AECs.

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Definition

An abstract elementary class (AEC) is a concrete category **K** satisfying the following conditions:

- All morphisms are concrete monomorphisms (injections).
- **K** has concrete directed colimits (also known as direct limits – basically closure under unions of increasing chains).
- (Smallness condition) Every object is a directed colimit of a fixed set of “small” subobjects.

Examples of abstract elementary classes

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Linear orderings with initial segment embeddings are not an AEC (do not satisfy the smallness condition).

Any AEC is an accessible category: a category with all sufficiently directed colimits satisfying a certain smallness condition.

Abstract elementary classes and logic

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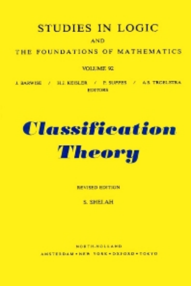
We will call such a category a *first-order class*. It is one of the basic objects of study in model theory.

Stability theory was developped for first-order classes first, by Saharon Shelah.



STUDIES IN LOGIC
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Classification Theory

REVISED EDITION
S. SHELAH

NORTH-HOLLAND
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Beyond first-order classes

There are some good reasons to look at more general classes. On the logic side, one can consider the infinitary logic $\mathbb{L}_{\infty,\omega}$, where infinite conjunctions and disjunctions are allowed (this logic also yields AECs, and usually any problem that is hard for AECs is hard already for this logic).

For example, we can say:

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Also, the morphisms of first-order classes are not so natural.

Eventual categoricity

Theorem (Morley, 1965)

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Morley's theorem was generalized to all first-order classes by Shelah (1974). He then asked about infinitary logics, and introduced AECs as a general framework to study the following question (*Shelah's eventual categoricity conjecture*).

Conjecture (Shelah, late seventies)

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One motivating goal is to develop stability theory for AECs.

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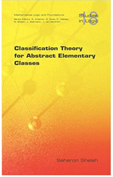
Partial approximations before my thesis include: Shelah 1983, Makkai-Shelah 1990, Shelah 1999, Shelah-Villaveces 1999, VanDieren 2006, Grossberg-VanDieren 2006, Shelah 2009, Hyttinen-Kesälä 2011, Boney 2014.

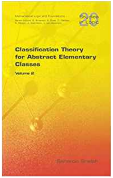


Mathematical Logic and Foundations

Studies in Logic 18

Classification Theory for Abstract Elementary Classes

Saharon Shelah



Mathematical Logic and Foundations

Studies in Logic 20

Classification Theory for Abstract Elementary Classes

Volume 2

Saharon Shelah

Toward Shelah's eventual categoricity conjecture

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Theorem (V. 2019)

Assuming a weakening of the GCH, Shelah's eventual categoricity conjecture is true for AECs with amalgamation. In this case one can list all possibilities for the class of cardinals in which the category has a unique object.

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Not all AECs are tame in general. Boney (2014) showed tameness follows from a large cardinal axiom, and always holds in universal AECs.

Still, at the time there was some doubt about how reasonable it was to assume tameness.

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Theorem (Kucera and Mazari-Armida)

The AEC of R-modules with pure embeddings is tame.

Some characterizations of stability

Theorem (V. 2016, Boney)

A tame AEC $\mathbf{K}$ with amalgamation is stable if and only if it does not have the "order property": any faithful functor $\mathbf{Lin} \xrightarrow{F} \mathbf{K}$ factors through the forgetful functor.

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Theorem (The stability spectrum theorem; V. 2018)

Assume SCH (a weakening of the GCH). If **K** is a stable tame AEC with amalgamation, then there exists a cardinal γ such that for all high-enough λ, **K** is stable in λ if and only if $\lambda = \lambda^{<\gamma}$.

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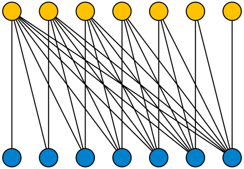
It is given by a half graph: for any linear ordering L, consider the bipartite graph on $L \sqcup L$ where we put an edge from i to j if only if $i \leq j$ (the picture below is for $L = \{1, 2, 3, 4, 5, 6, 7\}$):



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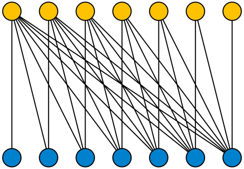

Graphs omitting half graphs are studied in finite combinatorics too (Malliaris-Shelah, *Regularity lemmas for stable graphs*. TAMS 2014).

Stable independence

The proofs of the eventual categoricity conjecture, of the partition theorem $\lambda^+ \xrightarrow{\mathbf{K}} (\lambda^+)^{<\omega}_{\lambda}$, and of the stability spectrum theorem involve describing what it means for a type to be “determined” over a small base. This is called forking in the first-order context, and is the key tool developped by Shelah in his classification theory book. It generalizes algebraic independence in fields.

Stable independence

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Unfortunately Shelah's definition is syntactic, hard to describe, and some properties depend on compactness. With my collaborators, we found a completely category-theoretic definition.

Definition (Equivalence of amalgam)

Consider a diagram: $B \leftarrow A \rightarrow C$

Two amalgams $B \rightarrow D^a \leftarrow C$, $B \rightarrow D^b \leftarrow C$ of this diagram are *equivalent* if there exists D and arrows making the following diagram commute:

$$\begin{tikzcd}
 & D^b & \\
B \arrow[ur] \arrow[rr] & & D^a \\
A \arrow[u] \arrow[r] & C \arrow[uu] \arrow[ur] &
\end{tikzcd}$$

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Example: in $\mathbf{Set}_{mono}$, $\{0\}$ and $\{1\}$ have two non-equivalent amalgams over $\emptyset$: $\{0, 1\}$ and $\{1\}$ (with the expected morphisms).

Definition (Stable independence; Lieberman-Rosický-V., 2019)

A *stable independence notion* is a class of squares (called *independent squares*, marked with $\mathop{\smile}\limits^{|}$) such that:

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4. Symmetry:

$$\begin{tikzcd} B \arrow[r] & D \\ A \arrow[u] \arrow[r] \arrow[ur, phantom, "\perp"] & C \arrow[u] \end{tikzcd} \Rightarrow \begin{tikzcd} C \arrow[r] & D \\ A \arrow[u] \arrow[r] \arrow[ur, phantom, "\perp"] & B \arrow[u] \end{tikzcd}$$

Definition (stable independence notion - continued)

5. Transitivity:

$$\begin{tikzcd}
B \arrow[r] & D \arrow[r] & F & & B \arrow[r] & {} \arrow[r] & F \\
A \arrow[u] \arrow[r] \arrow[ur, phantom, "\perp"] & C \arrow[u] \arrow[r] \arrow[ur, phantom, "\perp"] & E \arrow[u] & \Rightarrow & A \arrow[u] \arrow[r] & {} \arrow[r] \arrow[u, phantom, "\perp"] & E \arrow[u]
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6. Accessibility: the category whose objects are arrows and whose morphisms are independent squares is accessible. This implies that any arrow can be "filtered" in an independent way:

$$\begin{tikzcd}
M \arrow[r] & N \\
M_i \arrow[u, dotted] \arrow[r, dotted] \arrow[ur, phantom, "\perp"] & N_i \arrow[u, dotted]
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Theorem (Canonicity theorem; Lieberman-Rosický-V. 2019)

A category with directed colimits (in particular an AEC) has *at most one* stable independence notion.

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Theorem (Lieberman-Rosický-V. 2019)

An AEC with a stable independence notion has amalgamation, is tame, and is stable.

Certain converses are true too (for example in first-order classes, or assuming large cardinals).

Stable independence and cofibrant generation

Theorem (Lieberman-Rosický-V.)

Let $\mathcal{K}$ be an accessible, bicomplete category (like the category of R-modules with homomorphisms). Let $\mathcal{M}$ be a class of morphisms of $\mathcal{K}$ such that:

1. $\mathcal{M}$ contains all isomorphisms, is closed under transfinite compositions, pushouts, and retracts.
2. The induced category $\mathcal{K}_{\mathcal{M}}$ is accessible and closed under directed colimits in $\mathcal{K}$.
3. $\mathcal{M}$ is coherent: if $A \xrightarrow{f} B \xrightarrow{g} C$, $g, gf \in \mathcal{M}$, then $f \in \mathcal{M}$.

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Then $\mathcal{K}_{\mathcal{M}}$ has stable independence if and only if $\mathcal{M}$ is cofibrantly generated (i.e. can be generated from a subset using transfinite compositions, pushouts, and retracts).

New examples of stable independence

Corollary (Lieberman-Rosický-V.)

1. The AEC of flat R-modules with flat morphisms (more generally, any AEC of "roots of Ext") has stable independence.

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2. Any Grothendieck topos restricted to regular monos has stable independence.
3. Any Grothendieck abelian category restricted to monos has stable independence.
4. Any combinatorial model category where all objects are cofibrant and whose cofibrations are coherent (e.g. monos) has stable independence, when restricted to its cofibrations.

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The category-theoretic definition of stable independence also naturally yields higher-dimensional generalizations (independent cubes). These are well known in model theory but the earlier definitions are ad-hoc and complicated. The goal is now to *develop a systematic theory, and also to find more examples.*

Thank you!

Some references:

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