Stability theory for concrete categories

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We will work combinatorially: a category where there are few ways to add points will be called *stable*.

(this is not related to other meanings of stability in mathematics)

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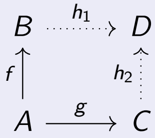
Definition

Given a concrete category **K** with amalgamation and an object A of **K**, a *type over* A is just a pair $(x, A \xrightarrow{f} B)$, with $x \in B$. Two types $(x, A \xrightarrow{f} B)$, $(y, A \xrightarrow{g} C)$ are considered *the same* if there exists maps h_1, h_2 so that $h_1(x) = h_2(y)$ and the following diagram commutes:



Types in fields, linear orders, and graphs

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If we restrict to graphs with finite degrees, we obtain again at most $\max(|V(G)|, \aleph_0)$ types over G

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- (Eklof 1971, Mazari-Armida) The category of R-modules with embeddings is always stable, and superstable if and only if R is Noetherian.
- (Kucera and Mazari-Armida) The category of R-modules with pure embeddings is always stable, and superstable if and only if R is pure-semisimple.

Abstract elementary classes: a framework for stability theory

Definition (Shelah, late 1970s)

An abstract elementary class (AEC) is a concrete category **K** satisfying the following conditions:

- All morphisms are concrete monomorphisms (injections).
- **K** has concrete directed colimits (also known as direct limits – basically closure under unions of increasing chains).
- (Smallness condition) Every object is a directed colimit of a fixed set of “small” subobjects.

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Any AEC is an accessible category: a category with all sufficiently directed colimits satisfying a certain smallness condition.

Abstract elementary classes and logic

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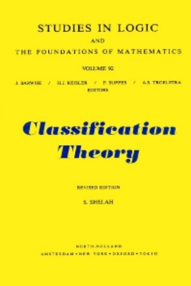
We will call such a category a *first-order class*. It is one of the basic objects of study in model theory.

Stability theory was developped for first-order classes first, by Saharon Shelah.



Beyond first-order classes

There are some good reasons to look at more general classes. On the logic side, one can consider the infinitary logic $\mathbb{L}_{\infty,\omega}$, where infinite conjunctions and disjunctions are allowed (this logic also yields AECs, and usually any problem that is hard for AECs is hard already for this logic).

For example, we can say:

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First-order classes are important, because of the *compactness theorem*: if all finite subsets of a given theory have a model, then the whole theory has a model. This is powerful (one can use it to build models for nonstandard analysis) but means that many interesting categories are not first-order.

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Also, the *morphisms* of first-order classes are not so natural.

Examples

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- However the category of fields is not: while the axioms of fields are first-order, the embedding $\mathbb{Q} \to \mathbb{R}$ does not preserve all formulas (consider $(\exists x)(x \cdot x = 2)$).
- In fact none of the other examples given so far are first-order.

Eventual categoricity: a test question

Theorem (Morley, 1965)

A countable set of first-order sentences with a single model of *some* uncountable cardinality has a single model in *all* uncountable cardinalities.

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Morley's theorem was generalized to all first-order classes by Shelah (1974). He then asked about infinitary logics, and introduced AECs as a general framework to study the following question (*Shelah's eventual categoricity conjecture*).

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One motivating goal is to develop stability theory for AECs.

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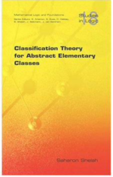
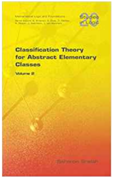
Partial approximations before my thesis include: Shelah 1983, Makkai-Shelah 1990, Shelah 1999, Shelah-Villaveces 1999, VanDieren 2006, Grossberg-VanDieren 2006, Shelah 2009, Hyttinen-Kesälä 2011, Boney 2014.





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Theorem (V. 2019)

Assuming the GCH, Shelah's eventual categoricity conjecture is true for AECs with amalgamation. In this case one can list all possibilities for the class of cardinals in which the category has a unique object.

A characterization of stability

Theorem (V. 2016, Boney)

A tame AEC $\mathbf{K}$ with amalgamation is stable if and only if it does not have the "order property": any faithful functor $\mathbf{Lin} \xrightarrow{F} \mathbf{K}$ factors through the forgetful functor.

$$
\begin{array}{ccc}
\mathbf{Lin} & \xrightarrow{F} & \mathbf{K} \\
\Big\downarrow U & \nearrow & \\
\mathbf{Set} & &
\end{array}
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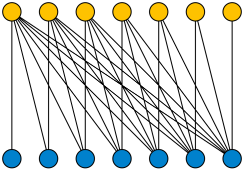
It is given by a half graph: for any linear ordering L, consider the bipartite graph on $L \sqcup L$ where we put an edge from i to j if only if $i \leq j$ (the picture below is for $L = \{1, 2, 3, 4, 5, 6, 7\}$):



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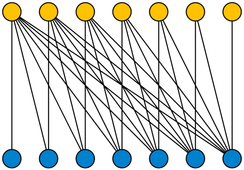

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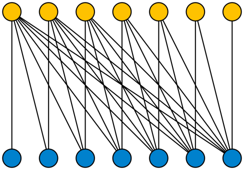

Graphs omitting half graphs are studied in finite combinatorics too (Malliaris-Shelah, *Regularity lemmas for stable graphs*. TAMS 2014). One can also show stability is equivalent to a certain "relativized" infinite Ramsey theorem. Thus in a sense *stability studies "tame" universes for (finite or infinite) combinatorics.*

Stable independence

The proofs of the eventual categoricity conjecture and of many results in stability theory involve describing what it means for a type to be "determined" over a small base. This is called forking in the first-order context, and is the key tool developped by Shelah in his classification theory book. It generalizes algebraic independence in fields.

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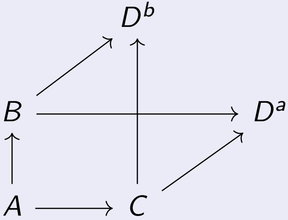
Unfortunately Shelah’s definition is syntactic, hard to describe, and some properties depend on compactness. With my collaborators, we found a completely category-theoretic definition.

Definition (Equivalence of amalgam)

Consider a diagram: $B \leftarrow A \rightarrow C$

Two amalgams $B \rightarrow D^a \leftarrow C$, $B \rightarrow D^b \leftarrow C$ of this diagram are *equivalent* if there exists D and arrows making the following diagram commute:



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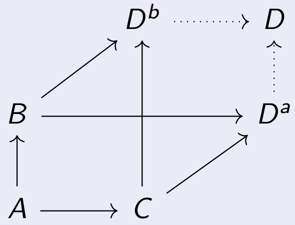

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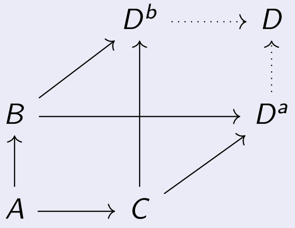

Example: in $\mathbf{Set}_{mono}$, $\{0\}$ and $\{1\}$ have two non-equivalent amalgams over $\emptyset$: $\{0, 1\}$ and $\{1\}$ (with the expected morphisms).

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4. Symmetry:

$$\begin{array}{ccc} B & \longrightarrow & D \\ \uparrow & \downarrow & \uparrow \\ A & \longrightarrow & C \end{array} \quad \Rightarrow \quad \begin{array}{ccc} C & \longrightarrow & D \\ \uparrow & \downarrow & \uparrow \\ A & \longrightarrow & B \end{array}$$

Definition (stable independence notion - continued)

5. Transitivity:

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6. Accessibility: the category whose objects are arrows and whose morphisms are independent squares is accessible. This implies that any arrow can be "filtered" in an independent way:

$$
\begin{array}{ccc}
M & \longrightarrow & N \\
\vdots\uparrow & \overset{|}{\smile} & \vdots\uparrow \\
M_i & \cdots\cdots\rightarrow & N_i
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Theorem (Lieberman-Rosický-V. 2019)

An AEC with a stable independence notion has amalgamation, is tame, and is stable.

Certain converses are true too (for example in first-order classes, or assuming large cardinals).

Stable independence and cofibrant generation

Theorem (Lieberman-Rosický-V.)

Let $\mathcal{K}$ be an accessible, bicomplete category (like the category of R-modules with homomorphisms). Let $\mathcal{M}$ be a class of morphisms of $\mathcal{K}$ such that:

1. $\mathcal{M}$ contains all isomorphisms, is closed under transfinite compositions, pushouts, and retracts.
2. The induced category $\mathcal{K}_{\mathcal{M}}$ is accessible and closed under directed colimits in $\mathcal{K}$.
3. $\mathcal{M}$ is coherent: if $A \xrightarrow{f} B \xrightarrow{g} C$, $g, gf \in \mathcal{M}$, then $f \in \mathcal{M}$.

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Then $\mathcal{K}_{\mathcal{M}}$ has stable independence if and only if $\mathcal{M}$ is cofibrantly generated (i.e. can be generated from a subset using transfinite compositions, pushouts, and retracts).

New examples of stable independence

Corollary (Lieberman-Rosický-V.)

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3. Any Grothendieck abelian category restricted to monos has stable independence.
4. Any Cisinski model category restricted to monos has stable independence.

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In hindsight, it seems this was too restrictive: many interesting categories are not first-order but still admit stability theory.

Summary and future work

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The notion of stable independence is central to stability theory. The category-theoretic definition is simple, and it seems it should appear in more places: *where else can we find it?*

Summary and future work

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The category-theoretic definition of stable independence also naturally yields higher-dimensional generalizations (independent cubes). These are well known in model theory but the earlier definitions are ad-hoc and complicated. The goal is now to *develop a systematic theory, and also to find more examples.*

Thank you!

Some references:

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