Indiscernible extraction and Morley sequences

Sebastien Vasey

Carnegie Mellon University

July 19, 2014
Logic Colloquium 2014
Vienna University of Technology

Main results

In ZFC *minus replacement*:

Theorem

Let T be a *simple* first-order theory. Let $M \models T$ and let $A \subseteq B \subseteq |M|$ be sets. Let $p \in S(B)$ be a type that does not fork over A. Then (inside some elementary extension of M) there is a Morley sequence $\langle \bar{b}_i \mid i < \omega \rangle$ for p over A.

Main results

In ZFC *minus replacement*:

Theorem

Let T be a *simple* first-order theory. Let $M \models T$ and let $A \subseteq B \subseteq |M|$ be sets. Let $p \in S(B)$ be a type that does not fork over A. Then (inside some elementary extension of M) there is a Morley sequence $\langle \bar{b}_i \mid i < \omega \rangle$ for p over A.

Corollary (Independently proven by Tsuboi)

In simple theories, forking is the same as dividing.

Main results

In ZFC *minus replacement*:

Theorem

Let T be a *simple* first-order theory. Let $M \models T$ and let $A \subseteq B \subseteq |M|$ be sets. Let $p \in S(B)$ be a type that does not fork over A. Then (inside some elementary extension of M) there is a Morley sequence $\langle \bar{b}_i \mid i < \omega \rangle$ for p over A.

Corollary (Independently proven by Tsuboi)

In simple theories, forking is the same as dividing.

- In ZFC both results are well known, but we give a new proof that uses only axioms from "ordinary" mathematics.

Main results

In ZFC *minus replacement*:

Theorem

Let T be a *simple* first-order theory. Let $M \models T$ and let $A \subseteq B \subseteq |M|$ be sets. Let $p \in S(B)$ be a type that does not fork over A. Then (inside some elementary extension of M) there is a Morley sequence $\langle \bar{b}_i \mid i < \omega \rangle$ for p over A.

Corollary (Independently proven by Tsuboi)

In simple theories, forking is the same as dividing.

- In ZFC both results are well known, but we give a new proof that uses only axioms from "ordinary" mathematics.
- This answers questions of Baldwin and Grossberg, Iovino, Lessmann.

Why ZFC minus replacement?

- We want to avoid using "big" cardinals like $\beth_{(2^{|T|})^+}$ (they are rarely used when the theory is *stable*).

Why ZFC minus replacement?

- We want to avoid using "big" cardinals like $\beth_{(2^{|T|})^+}$ (they are rarely used when the theory is *stable*).
- The proofs usually give more information.

Why ZFC minus replacement?

- We want to avoid using "big" cardinals like $\beth_{(2^{|T|})^+}$ (they are rarely used when the theory is *stable*).
- The proofs usually give more information.
- In our case, we obtain a new characterization of simplicity in terms of definability of forking (pointed out by Kaplan).

Why ZFC minus replacement?

- We want to avoid using "big" cardinals like $\beth_{(2^{|T|})^+}$ (they are rarely used when the theory is *stable*).
- The proofs usually give more information.
- In our case, we obtain a new characterization of simplicity in terms of definability of forking (pointed out by Kaplan).
- Harnik's work on the reverse mathematics of stability theory.

Why ZFC minus replacement?

- We want to avoid using "big" cardinals like $\beth_{(2^{|T|})^+}$ (they are rarely used when the theory is *stable*).
- The proofs usually give more information.
- In our case, we obtain a new characterization of simplicity in terms of definability of forking (pointed out by Kaplan).
- Harnik's work on the reverse mathematics of stability theory.

- However, for convenience only, we will work inside a big saturated-enough monster model of a fixed first-order theory T.

Independent and Morley sequences

Definition

Let $\mathbf{J} := \langle \bar{a}_j \mid j < \alpha \rangle$ be a sequence of finite tuples of the same arity. Let $A \subseteq B$ be sets, and let $p \in S(B)$ be a type that does not fork over A.

$\mathbf{J}$ is said to be an *independent sequence for* p *over* A if:

Independent and Morley sequences

Definition

Let $\mathbf{J} := \langle \bar{a}_j \mid j < \alpha \rangle$ be a sequence of finite tuples of the same arity. Let $A \subseteq B$ be sets, and let $p \in S(B)$ be a type that does not fork over A.

$\mathbf{J}$ is said to be an *independent sequence for p over A* if:

1. For all $j < \alpha$, $\bar{a}_j \models p$.

Independent and Morley sequences

Definition

Let $\mathbf{J} := \langle \bar{a}_j \mid j < \alpha \rangle$ be a sequence of finite tuples of the same arity. Let $A \subseteq B$ be sets, and let $p \in S(B)$ be a type that does not fork over A.

$\mathbf{J}$ is said to be an *independent sequence for p over A* if:

1. For all $j < \alpha$, $\bar{a}_j \models p$.
2. For all $j < \alpha$, $\text{tp}(\bar{a}_j / B \cup \{\bar{a}_{j'} \mid j' < j\})$ does not fork over A.

Independent and Morley sequences

Definition

Let $\mathbf{J} := \langle \bar{a}_j \mid j < \alpha \rangle$ be a sequence of finite tuples of the same arity. Let $A \subseteq B$ be sets, and let $p \in S(B)$ be a type that does not fork over A.

$\mathbf{J}$ is said to be an *independent sequence for p over A* if:

1. For all $j < \alpha$, $\bar{a}_j \models p$.
2. For all $j < \alpha$, $\text{tp}(\bar{a}_j / B \cup \{\bar{a}_{j'} \mid j' < j\})$ does not fork over A.

$\mathbf{J}$ is said to be a *Morley sequence for p over A* if:

Independent and Morley sequences

Definition

Let $\mathbf{J} := \langle \bar{a}_j \mid j < \alpha \rangle$ be a sequence of finite tuples of the same arity. Let $A \subseteq B$ be sets, and let $p \in S(B)$ be a type that does not fork over A.

$\mathbf{J}$ is said to be an *independent sequence for p over A* if:

1. For all $j < \alpha$, $\bar{a}_j \models p$.
2. For all $j < \alpha$, $\text{tp}(\bar{a}_j / B \cup \{\bar{a}_{j'} \mid j' < j\})$ does not fork over A.

$\mathbf{J}$ is said to be a *Morley sequence for p over A* if:

1. $\mathbf{J}$ is an independent sequence for p over A.

Independent and Morley sequences

Definition

Let $\mathbf{J} := \langle \bar{a}_j \mid j < \alpha \rangle$ be a sequence of finite tuples of the same arity. Let $A \subseteq B$ be sets, and let $p \in S(B)$ be a type that does not fork over A.

$\mathbf{J}$ is said to be an *independent sequence for p over A* if:

1. For all $j < \alpha$, $\bar{a}_j \models p$.
2. For all $j < \alpha$, $\text{tp}(\bar{a}_j / B \cup \{\bar{a}_{j'} \mid j' < j\})$ does not fork over A.

$\mathbf{J}$ is said to be a *Morley sequence for p over A* if:

1. $\mathbf{J}$ is an independent sequence for p over A.
2. $\mathbf{J}$ is indiscernible over B.

Some easy remarks

- If p does not fork over A, we can build an independent sequence $\mathbf{J} := \langle \bar{a}_j \mid j < \alpha \rangle$ for p by repeated use of the extension property.

Some easy remarks

- If p does not fork over A, we can build an independent sequence $\mathbf{J} := \langle \bar{a}_j \mid j < \alpha \rangle$ for p by repeated use of the extension property.
- If T is stable and $\alpha \geq \left(2^{|T|}\right)^+$, we can then find a subsequence of $\mathbf{J}$ which is indiscernible, and hence Morley.

Some easy remarks

- If p does not fork over A, we can build an independent sequence $\mathbf{J} := \langle \bar{a}_j \mid j < \alpha \rangle$ for p by repeated use of the extension property.
- If T is stable and $\alpha \geq \left(2^{|T|}\right)^+$, we can then find a subsequence of $\mathbf{J}$ which is indiscernible, and hence Morley.
- If T is unstable, there need not be an indiscernible subsequence. But we can still build indiscernibles "on the side":

Fact (The indiscernible extraction theorem)

Let B be a set. Let $\mu := \beth_{\left(2^{|T|+|B|}\right)^+}$, and let $\langle \bar{a}_j \mid j < \mu \rangle$ be a sequence of finite tuples. Then there exists a sequence $\left\langle \bar{b}_i \mid i < \omega \right\rangle$, indiscernible over B such that:
For any $i_0 < \ldots < i_{n-1} < \omega$, there exists $j_0 < \ldots < j_{n-1} < \mu$ so that $\text{tp}(\bar{b}_{i_0} \ldots \bar{b}_{i_{n-1}}/B) = \text{tp}(\bar{a}_{j_0} \ldots \bar{a}_{j_{n-1}}/B)$.

Fact (The indiscernible extraction theorem)

Let B be a set. Let $\mu := \beth_{\left(2^{|T|+|B|}\right)^+}$, and let $\langle \bar{a}_j \mid j < \mu \rangle$ be a sequence of finite tuples. Then there exists a sequence $\left\langle \bar{b}_i \mid i < \omega \right\rangle$, indiscernible over B such that:
For any $i_0 < \ldots < i_{n-1} < \omega$, there exists $j_0 < \ldots < j_{n-1} < \mu$ so that $\text{tp}(\bar{b}_{i_0} \ldots \bar{b}_{i_{n-1}}/B) = \text{tp}(\bar{a}_{j_0} \ldots \bar{a}_{j_{n-1}}/B)$.

Using invariance and finite character of forking, it is easy to argue that if $\langle \bar{a}_j \mid j < \mu \rangle$ is independent, then $\left\langle \bar{b}_i \mid i < \omega \right\rangle$ also is independent (and so is Morley).

$\beth_{\left(2^{|T|+|B|}\right)^+}$ is too much, so we will use the following weak version that works for ω:

Fact (The weak indiscernible extraction theorem)

Let B be a set. Let $\langle \bar{a}_j \mid j < \omega \rangle$ be a sequence of finite tuples. Then there exists a sequence $\left\langle \bar{b}_i \mid i < \omega \right\rangle$, indiscernible over B such that:
For any $i_0 < \ldots < i_{n-1} < \omega$, for all *finite* $q \subseteq \text{tp}(\bar{b}_{i_0} \ldots \bar{b}_{i_{n-1}} / B)$, there exists $j_0 < \ldots < j_{n-1} < \omega$ so that $\bar{a}_{j_0} \ldots \bar{a}_{j_{n-1}} \models q$.

$\beth_{(2^{|T|+|B|})^+}$ is too much, so we will use the following weak version that works for ω:

Fact (The weak indiscernible extraction theorem)

Let B be a set. Let $\langle \bar{a}_j \mid j < \omega \rangle$ be a sequence of finite tuples. Then there exists a sequence $\left\langle \bar{b}_i \mid i < \omega \right\rangle$, indiscernible over B such that:
For any $i_0 < \ldots < i_{n-1} < \omega$, for all *finite* $q \subseteq \operatorname{tp}(\bar{b}_{i_0} \ldots \bar{b}_{i_{n-1}}/B)$, there exists $j_0 < \ldots < j_{n-1} < \omega$ so that $\bar{a}_{j_0} \ldots \bar{a}_{j_{n-1}} \models q$.

However this does *not* give us enough invariance to deduce that independence of $\langle \bar{a}_j \mid j < \omega \rangle$ implies independence of $\left\langle \bar{b}_i \mid i < \omega \right\rangle$.

Dual finite character

To get the desired conclusion, we will assume the following local definability property of forking:

Definition

Forking is said to have *dual finite character (DFC)* if whenever $\text{tp}(\bar{c}/A\bar{b})$ forks over A, there is a formula $\phi(\bar{x}, \bar{y})$ over A such that:

- $\models \phi[\bar{c}, \bar{b}]$, and:
- $\models \phi[\bar{c}, \bar{b}']$ implies $\text{tp}(\bar{c}/A\bar{b}')$ forks over A.

Theorem

Assume forking has DFC. Let $A \subseteq B$ be sets. Let $p \in S(B)$ be a type that does not fork over A. Then there is a Morley sequence $\langle \bar{b}_i \mid i < \omega \rangle$ for p over A.

Proof sketch

1. Build an independent sequence $\langle \bar{a}_j \mid j < \omega \rangle$ for p over A.

Proof sketch

1. Build an independent sequence $\langle \bar{a}_j \mid j < \omega \rangle$ for p over A.
2. Use the weak indiscernible extraction theorem to obtain $\left\langle \bar{b}_i \mid i < \omega \right\rangle$ indiscernible over B such that any formula realized by the $\bar{b}_i$s is realized by some of the $\bar{a}_j$s. This is independent for p over A because:

Proof sketch

1. Build an independent sequence $\langle \bar{a}_j \mid j < \omega \rangle$ for p over A.
2. Use the weak indiscernible extraction theorem to obtain $\left\langle \bar{b}_i \mid i < \omega \right\rangle$ indiscernible over B such that any formula realized by the $\bar{b}_i$s is realized by some of the $\bar{a}_j$s. This is independent for p over A because:
   1. For all $i < \omega$, $\bar{b}_i$ realizes p: if not, take a formula witnessing it and deduce that some $\bar{a}_j$ does not realize p.

Proof sketch

1. Build an independent sequence $\langle \bar{a}_j \mid j < \omega \rangle$ for p over A.
2. Use the weak indiscernible extraction theorem to obtain $\left\langle \bar{b}_i \mid i < \omega \right\rangle$ indiscernible over B such that any formula realized by the $\bar{b}_i$s is realized by some of the $\bar{a}_j$s. This is independent for p over A because:
   1. For all $i < \omega$, $\bar{b}_i$ realizes p: if not, take a formula witnessing it and deduce that some $\bar{a}_j$ does not realize p.
   2. For all $i < \omega$, $\text{tp}(\bar{b}_i / B \cup \{\bar{b}_{i'} \mid i' < i\})$ does not fork over A:

Proof sketch

1. Build an independent sequence $\langle \bar{a}_j \mid j < \omega \rangle$ for p over A.
2. Use the weak indiscernible extraction theorem to obtain $\left\langle \bar{b}_i \mid i < \omega \right\rangle$ indiscernible over B such that any formula realized by the $\bar{b}_i$s is realized by some of the $\bar{a}_j$s. This is independent for p over A because:
    1. For all $i < \omega$, $\bar{b}_i$ realizes p: if not, take a formula witnessing it and deduce that some $\bar{a}_j$ does not realize p.
    2. For all $i < \omega$, $\text{tp}(\bar{b}_i / B \cup \{\bar{b}_{i'} \mid i' < i\})$ does not fork over A:
        1. If not, let $\phi(\bar{x}, \bar{y}_0 \ldots \bar{y}_{n-1})$ be as given by DFC.

Proof sketch

1. Build an independent sequence $\langle \bar{a}_j \mid j < \omega \rangle$ for p over A.
2. Use the weak indiscernible extraction theorem to obtain $\left\langle \bar{b}_i \mid i < \omega \right\rangle$ indiscernible over B such that any formula realized by the $\bar{b}_i$s is realized by some of the $\bar{a}_j$s. This is independent for p over A because:
    1. For all $i < \omega$, $\bar{b}_i$ realizes p: if not, take a formula witnessing it and deduce that some $\bar{a}_j$ does not realize p.
    2. For all $i < \omega$, $\text{tp}(\bar{b}_i / B \cup \{\bar{b}_{i'} \mid i' < i\})$ does not fork over A:
        1. If not, let $\phi(\bar{x}, \bar{y}_0 \ldots \bar{y}_{n-1})$ be as given by DFC.
        2. Find $\bar{a}_j$, $\bar{a}_{j_0} \ldots \bar{a}_{j_{n-1}}$ realizing ϕ.

Proof sketch

1. Build an independent sequence $\langle \bar{a}_j \mid j < \omega \rangle$ for p over A.
2. Use the weak indiscernible extraction theorem to obtain $\left\langle \bar{b}_i \mid i < \omega \right\rangle$ indiscernible over B such that any formula realized by the $\bar{b}_i$s is realized by some of the $\bar{a}_j$s. This is independent for p over A because:
    1. For all $i < \omega$, $\bar{b}_i$ realizes p: if not, take a formula witnessing it and deduce that some $\bar{a}_j$ does not realize p.
    2. For all $i < \omega$, $\text{tp}(\bar{b}_i / B \cup \{\bar{b}_{i'} \mid i' < i\})$ does not fork over A:
        1. If not, let $\phi(\bar{x}, \bar{y}_0 \ldots \bar{y}_{n-1})$ be as given by DFC.
        2. Find $\bar{a}_j$, $\bar{a}_{j_0} \ldots \bar{a}_{j_{n-1}}$ realizing ϕ.
        3. Use the definition of ϕ together with $\text{tp}(\bar{a}_j / B) = p = \text{tp}(\bar{b}_i / B)$ to see that $\text{tp}(\bar{a}_j / B \cup \{\bar{a}_{j'} \mid j' < j\})$ forks over A, contradiction.

When does forking have DFC?

Definition

Forking has the *symmetry property* when $\text{tp}(\bar{a}/A\bar{b})$ does not fork over A if and only if $\text{tp}(\bar{b}/A\bar{a})$ does not fork over A.

Proposition

If forking has the symmetry property, then it has DFC.

When does forking have DFC?

Definition

Forking has the *symmetry property* when $\mathrm{tp}(\bar{a}/A\bar{b})$ does not fork over A if and only if $\mathrm{tp}(\bar{b}/A\bar{a})$ does not fork over A.

Proposition

If forking has the symmetry property, then it has DFC.

Fact (Kim)

T is simple if and only if forking has the symmetry property.

When does forking have DFC?

Definition

Forking has the *symmetry property* when $\text{tp}(\bar{a}/A\bar{b})$ does not fork over A if and only if $\text{tp}(\bar{b}/A\bar{a})$ does not fork over A.

Proposition

If forking has the symmetry property, then it has DFC.

Fact (Kim)

T is simple if and only if forking has the symmetry property.

There is no circularity: methods of Adler can be used to prove this in ZFC minus replacement without relying on existence of Morley sequences.

Corollary

Assume T is simple. Let $A \subseteq B$ be sets. Let $p \in S(B)$ be a type that does not fork over A. Then there is a Morley sequence $\langle \bar{b}_i \mid i < \omega \rangle$ for p over A.

Proof: By Kim's theorem, forking has symmetry, and hence by the previous proposition has DFC. Apply the previous result.

Is DFC weaker than simplicity?

Is DFC weaker than simplicity?

No (Itay Kaplan, personal communication).

Is DFC weaker than simplicity?

No (Itay Kaplan, personal communication). The key is that symmetry fails *very badly* in nonsimple theories:

Fact (Chernikov)

Assume T is *not* simple. Then there is a model M and tuples $\bar{b}, \bar{c}$ such that $\text{tp}(\bar{b}/M\bar{c})$ is finitely satisfiable in M, but $\text{tp}(\bar{c}/M\bar{b})$ forks over M.

Is DFC weaker than simplicity?

No (Itay Kaplan, personal communication). The key is that symmetry fails *very badly* in nonsimple theories:

Fact (Chernikov)

Assume T is *not* simple. Then there is a model M and tuples $\bar{b}, \bar{c}$ such that $\text{tp}(\bar{b}/M\bar{c})$ is finitely satisfiable in M, but $\text{tp}(\bar{c}/M\bar{b})$ forks over M.

Corollary

T is simple if and only if forking has DFC.

Proof that DFC implies simple

We show the contrapositive. Assume T is not simple. Fix M, $\bar{b}$, $\bar{c}$ such that $\text{tp}(\bar{b}/M\bar{c})$ is finitely satisfiable in M, but $p := \text{tp}(\bar{c}/M\bar{b})$ forks over M. Assume $\phi(\bar{x}, \bar{b})$ is a formula over M in p.

Proof that DFC implies simple

We show the contrapositive. Assume T is not simple. Fix M, $\bar{b}$, $\bar{c}$ such that $\text{tp}(\bar{b}/M\bar{c})$ is finitely satisfiable in M, but $p := \text{tp}(\bar{c}/M\bar{b})$ forks over M. Assume $\phi(\bar{x}, \bar{b})$ is a formula over M in p.

1. By finite satisfiability, there is $\bar{b}' \in M$ such that $\models \phi[\bar{c}, \bar{b}']$.

Proof that DFC implies simple

We show the contrapositive. Assume T is not simple. Fix M, $\bar{b}$, $\bar{c}$ such that $\text{tp}(\bar{b}/M\bar{c})$ is finitely satisfiable in M, but $p := \text{tp}(\bar{c}/M\bar{b})$ forks over M. Assume $\phi(\bar{x}, \bar{b})$ is a formula over M in p.

1. By finite satisfiability, there is $\bar{b}' \in M$ such that $\models \phi[\bar{c}, \bar{b}']$.
2. So $\text{tp}(\bar{c}/M\bar{b}') = \text{tp}(\bar{c}/M)$ is finitely satisfiable in M and hence does not fork over M.

Proof that DFC implies simple

We show the contrapositive. Assume T is not simple. Fix M, $\bar{b}$, $\bar{c}$ such that $\text{tp}(\bar{b}/M\bar{c})$ is finitely satisfiable in M, but $p := \text{tp}(\bar{c}/M\bar{b})$ forks over M. Assume $\phi(\bar{x}, \bar{b})$ is a formula over M in p.

1. By finite satisfiability, there is $\bar{b}' \in M$ such that $\models \phi[\bar{c}, \bar{b}']$.
2. So $\text{tp}(\bar{c}/M\bar{b}') = \text{tp}(\bar{c}/M)$ is finitely satisfiable in M and hence does not fork over M.
3. So ϕ cannot witness DFC.

Thank you!

- For further reference, see:
  Sebastien Vasey, *Indiscernible extraction and Morley sequences*, Accepted (June 9, 2014), Notre Dame Journal of Formal Logic.
- A preprint can be accessed from my webpage:
  `http://math.cmu.edu/~svasey/`
- For a direct link, you can take a picture of the QR code below: