SOLVING LINEAR DIFFERENCE EQUATIONS

A *linear difference equation* is an equation of the form:

$$\alpha_k x_{n+k} + \alpha_{k-1} x_{n+k-1} + \ldots + \alpha_0 x_n = r_n$$

Where n ranges over all natural numbers, $\alpha_0, \ldots, \alpha_k$ are complex numbers (in this course real numbers), and r_n is a fixed function of n (in this course usually pretty simple, like a constant or a polynomial). We say that the equation has *order* k if $\alpha_k \neq 0$. A *solution* of the difference equation is a sequence $\langle x_n : n \in \mathbb{N} \rangle$ satisfying the equation for all $n \in \mathbb{N}$. Since a difference equation usually has many solutions, we may impose *boundary conditions* of the form $x_0 = \beta_0, \ldots, x_{k-1} = \beta_{k-1}$. We call a linear difference equation *homogeneous* if $r_n = 0$.

Example. Consider the homogeneous linear difference equation $x_{n+1} + 2x_n = 0$. Rearranging, we get that $x_{n+1} = -2x_n$. Since (if $n \geq 1$) $x_n = -2x_{n-1}$, it is easy to guess that $x_n = (-2)^n A$ is a solution, for any real constant A. If we specify the *boundary condition* that $x_0 = 1$, we get that $A = 1$, so the solution is $x_n = (-2)^n$. It can be shown that this is the unique solution.

Remark. A linear difference equation can be seen as an approximation to a linear *differential* equation. For example, the linear differential equation $x' = 0$ is the same as $0 = \lim_{h \to 0} \frac{x(t+h)-x(t)}{h}$. Taking $h \neq 0$ fixed and small, this equation is approximately the same as $0 = x(t+h) - x(t)$. Now let y_n be $x(t+nh)$. Then the last equation is $0 = y_{n+1} - y_n$, a linear difference equation.

Solving homogeneous linear difference equation. We want to solve $\alpha_k x_{n+k} + \alpha_{k-1} x_{n+k-1} + \ldots + \alpha_0 x_n = 0$. Note that linear combinations of solutions are also solutions. Thus the solutions form a *vector space*. One can show that this space will have dimension k, where k is the order of the equation. To find the solutions, we look at the *characteristic polynomial* of the equation, defined by $\alpha_k \theta^k + \alpha_{k-1} \theta^{k-1} + \ldots + \alpha_0$. If θ is a root of the characteristic polynomial, then one can check that $x_n = \theta^n$ will be a solution to the equation. More generally, if θ has multiplicity m, then one can check that $n^\ell \theta^n$ will be solutions for any $\ell \leq m$ (to see this, think about what you get when taking the derivative of the characteristic polynomial ℓ times).

Assuming that all solutions have multiplicity one, a general solution will have the form $A_1 \theta_1^n + A_2 \theta_2^n + \ldots + A_k \theta_k^n$, where $A_1, \ldots, A_k$ are complex numbers. If for example $\theta_1 = \theta_2$ has multiplicity 2, we would also need a solution of the form $Bn\theta_1^n$. To obtain a single solution from the general solution, one would then need k boundary conditions.

Example. Consider the homogeneous linear difference equation $-x_{n+2} + x_{n+1} + 3x_n = 0$. Set $x = \theta_n$ (θ nonzero) and solve for θ. We get $-\theta^{n+2} + \theta^{n+1} + 3\theta^n = 0$. Dividing out by θ^n, we get $-\theta^2 + \theta + 3 = 0$. This equation has solutions $\theta = \frac{1 \pm \sqrt{13}}{2}$. Thus the general solution of the homogeneous linear difference equation is of the form $A_1 \theta_1^n + A_2 \theta_2^n$, where $\theta_1 = \frac{1+\sqrt{13}}{2}$, $\theta_2 = \frac{1-\sqrt{13}}{2}$. Under the initial conditions

$x_0 = 0$ and $x_1 = 1$, we can solve the equations $A_1 + A_2 = 0$ and $A_1\theta_1 + A_2\theta_2 = 1$ to find A_1 and A_2.

Example. Consider the homogeneous linear difference equation $-x_{n+2} + 2x_{n+1} - x_n = 0$. The characteristic polynomial $-\theta^2 + 2\theta - 1$ has the single root $\theta = 1$ of multiplicity 2. Thus the general solution is of the form $x_n = A_1 + A_2 n$.

Solving linear difference equation. Consider the general case $\alpha_k x_{n+k} \alpha_{k-1} x_{n+k-1} + \ldots + \alpha_0 x_n = r_n$. Any solution will be of the form $x_n = y_n + s_n$, where $\langle y_n : n \in \mathbb{N} \rangle$ is a solution to the *homogeneous* version of the equation: $\alpha_k x_{n+k} \alpha_{k-1} x_{n+k-1} + \ldots + \alpha_0 x_n = 0$, and s_n is any solution to the original equation. In words, a solution to a linear difference equation is given by summing the solution to the homogeneous case to a particular solution. There is no general method to obtain the particular solution, one must guess it, but in simple cases it is not difficult.

Example. Consider the linear difference equation $-x_{n+2} + x_{n+1} + 3x_n = n$. We have found the homogeneous solutions y_n in the previous example. Let us now find a particular solution. Looking at the form of the right hand side, it seems plausible that the particular solution would look like $x_n = \alpha n + \beta$, where α and β are fixed complex numbers. Let us plug this into the equation. We get $n = -\alpha(n+2) - \beta + \alpha(n+1) + \beta + 3\alpha n + 3\beta = \alpha(3n-1) + 3\beta = 3\alpha n + 3\beta - \alpha$. Solving, we get $\alpha = \frac{1}{3}$, $\beta = \frac{1}{9}$. Thus the general solution[1] is given by $x_n = A_1\theta_1^n + A_2\theta_2^n + \frac{n}{3} + \frac{1}{9}$.

[1] One can check it is indeed *the* solution.