THE RANDOMIZED TWO ENVELOPES PROBLEM

Two envelopes each contain some amount of money. You know that one enveloppe contains more money than the other, but you do not know which, nor do you know the amounts involved. You are allowed to pick one envelope, look into it, and then either keep this envelope or switch and keep the other envelope. You would of course like a strategy to maximize the amount of money you obtain.

A simple strategy is to pick an envelope uniformly at random and keep it, regardless of what is inside it (after all, how can you know whether it is the higher or smaller amount?). If the two envelopes contain a and b dollars respectively, then this strategy yields an expected value of $\frac{a+b}{2}$ dollars.

Can one do better? The amazing answer is yes. We follow the following strategy:

(1) Choose one of the two envelopes uniformly at random.
(2) Look inside it, suppose the envelope contains Y dollars. Let X be an exponential random variable with parameter $\lambda > 0$ (the value of λ does not matter). If $X \leq Y$, keep the amount in the envelope. If $X > Y$, switch and pick the other envelope.

Let Z be the amount of money that you obtain using this strategy. Let a and b be the amount of money in each envelope, with $0 < a < b$. We want to compute $\mathbb{E}(Z)$. We will see that $\mathbb{E}(Z) > \frac{a+b}{2}$.

To compute $\mathbb{E}(Z)$, we condition on the value of Y. It can either be a or b. We have that:

$$\mathbb{E}(Z|Y=a) = aP(X \leq a) + bP(X > a) = a(1 - e^{-\lambda a}) + be^{-\lambda a} = a + e^{-\lambda a}(b-a)$$

Since we obtain a dollars if $X \leq Y = a$ and b dollars if $X > Y = a$. Similarly,

$$\mathbb{E}(Z|Y=b) = bP(X \leq b) + aP(X > b) = b(1 - e^{-\lambda b}) + ae^{-\lambda b} = b - e^{-\lambda b}(b-a)$$

Since Y has equal chances of being a or b, we obtain:

$$\mathbb{E}(Z) = \frac{1}{2}\mathbb{E}(Z|Y=a) + \frac{1}{2}\mathbb{E}(Z|Y=b) = \frac{a+b}{2} + \frac{b-a}{2}(e^{-\lambda a} - e^{-\lambda b})$$

Observe that $\frac{b-a}{2}$ is strictly positive, and since the exponential function is strictly monotonic, so is $e^{-\lambda a} - e^{-\lambda b}$. Thus $\frac{b-a}{2}(e^{-\lambda a} - e^{-\lambda b}) > 0$. We conclude that we indeed have $\mathbb{E}(Z) > \frac{a+b}{2}$.