# MATH 154 - PROBABILITY THEORY, SPRING 2018
# EXTRA CREDIT ASSIGNMENT

**Instructions.** The goal of this *optional* problem set is to guide you through the proof of the reconstruction an continuity theorems. It is designed so that most problems can be done individually of the others (of course taking previous ones as black boxes). This problem set can add up to 5% extra credit to your final grade. *If you choose to do this problem set, please do NOT look up the proofs in a textbook!*

**Due Friday, April 27, before 1PM.** Make sure to include your full name *and the list of your collaborators* (if any) with your assignment. You may discuss problems with others, but you may *not* keep a written record of your discussions. Please refer to the syllabus for details.

With regards to answering these problems, imagine that you are writing an answer to teach someone else in the class how to do the problem. In particular, you must give a complete outline for how you arrived at your answer. It is not sufficient to simply state a number or formula without providing the steps and reasoning that you used to produce the answer.

Everywhere below, "random variable" means "discrete or continuous random variable".

## 1. Proving the reconstruction theorem

(1) Show that $\int_a^b e^{it}dt = \frac{e^{ib}-e^{ia}}{i}$. *Hint: do NOT integrate $e^{it}$ as if it were a real-valued function.*

(2) Let $X$ be a random variable with distribution $F$ and characteristic function $\phi$. Let $a < b$ be real numbers and let $T > 0$. Show that:

$$\int \int_{-T}^{T} \left| \frac{e^{-ita} - e^{-itb}}{it} \phi(t) \right| dtdF \leq 2T(b-a)$$

*Hint: estimate $\frac{e^{-ita}-e^{-itb}}{it}$ using the previous part.*

(3) Show that $\lim_{T\to\infty} \int_0^T \frac{\sin(t)}{t} dt = \frac{\pi}{2}$. *Hint: use that $\frac{1}{t} = \int_0^\infty e^{-ut} du$ and recall the hint for the first extra credit problem of assignment 10.*

(4) Show that for any real number $\theta$,

$$\lim_{T\to\infty} \int_{-T}^{T} \frac{\sin(\theta t)}{t} dt = \pi \operatorname{sgn}(\theta)$$

where $\operatorname{sgn}(\theta) = 1$ if $\theta > 0$, $-1$ if $\theta < 0$, and $0$ if $\theta = 0$. Show in addition that there is a constant $M$ such that $\left|\int_{-T}^{T} \frac{\sin(\theta t)}{t} dt\right| < M$ for any real number $\theta$ and any $T \geq 0$.

(5) Let $X$ be a random variable with characteristic function $\phi$. Let $a < b$. Prove that:

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$$P(a < X < b) + \frac{P(X = a) + P(X = b)}{2} = \frac{1}{2\pi} \lim_{T\to\infty} \int_{-T}^{T} \frac{e^{-ita} - e^{-itb}}{it} \phi(t)dt$$

Deduce the reconstruction theorem (Fact 5.7 in the notes).

(6) Let $X$ be a random variable with characteristic function $\phi$. Assume that $\phi$ is absolutely integrable (i.e. $\int_{-\infty}^{\infty} |\phi(t)|dt < \infty$). Show that $X$ is continuous and that the probability density function of $X$ can be recovered from the *Fourier inversion formula*:

$$f_X(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \phi(t)dt$$

## 2. Proving the continuity theorem

(1) For $A$ a set of real numbers, let $\chi_A$ be the indicator function of $A$: $\chi_A : \mathbb{R} \to \mathbb{R}$ is defined by $\chi_A(x) = 1$ if $x \in A$ and $\chi_A(x) = 0$ if $x \notin A$. Let $X$ be a random variable with distribution function $F$.
   (a) Show that $F(a) = \int \chi_{(-\infty,a]} dF$.
   (b) Show that for $a < b$, $P(a < X \leq b) = \int \chi_{(a,b]} dF$.

(2) Let $X, X_1, X_2, \dots$ be random variables with distribution functions $F, F_1, F_2, \dots$. Let $f : \mathbb{R} \to \mathbb{C}$ be a continuous and bounded function (that is, there is $M$ such that $|f(x)| < M$ for all $x \in \mathbb{R}$). Assume that $X_n \xrightarrow{D} X$. Show that $\int f dF_n \to \int f dF$. Deduce (1) implies (2) in the continuity theorem (Fact 5.9 in the notes) *Hint: first do it for the indicator functions studied in the previous problem. Then approximate $f$ by such indicator functions and use the dominated convergence theorem.*

(3) Let $F : \mathbb{R} \to [0, 1]$ be a function satisfying the following properties:
   - $F$ is non-decreasing.
   - $\lim_{x\to-\infty} F(x) = 0$.
   - $\lim_{x\to\infty} F(x) = 1$.
   - $F$ is right-continuous. That is, $\lim_{x\downarrow a} F(x) = F(a)$ for any real number $a$.

   Prove that there exists a probability space $(\Omega, \mathcal{F}, P)$ and a random variable $X$ on that space such that $F$ is the distribution function of $X$. *Hint: Let $\Omega = \mathbb{R}$, $\mathcal{F}$ be the Borel subsets of $\mathbb{R}$, and $P((a, b]) = F(b) - F(a)$.*

(4) Let $F_1, F_2, \dots$ be a sequence of non-decreasing right-continuous functions. Assume that the sequence is bounded, i.e. there exists a constant $M$ such that $|F_n(x)| < M$ for any $n$ and $x$. Show that there exists a subsequence $(n_k)_{k\in\mathbb{N}}$ and a non-decreasing right-continuous function $F$ such that for any real number $x$ which is a continuity point of $F$, $F_{n_k}(x) \to F(x)$. *Hint: for any fixed $x$, you can use the Bolzano-Weierstrass theorem to get a convergence subsequence $F_{n_k}(x)$ going to some limit $F(x)$. Enumerate the rationals as $(x_m)_{m\in\mathbb{N}}$. Extract a subsequence $F_{n_k^0}(x_0)$ converging to some $F(x_0)$, then extract a subsequence $F_{n_k^1}$ of that which converges to some $F(x_1)$, etc. In the end, take $n_k = n_k^k$. Check that this works.*

(5) We say that a set $\{X_i : i \in I\}$ of random variables is *tight* if for any $\epsilon > 0$, there exists $a < b$ so that $P(a \leq X_n \leq b) \geq 1 - \epsilon$ for all $i \in I$. A sequence $X_1, X_2, \dots$ is *tight* if the corresponding set $\{X_1, X_2, \dots\}$ is tight.
  (a) Give an example of a set of random variables which is not tight.
  (b) Show that any subset of a tight set of random variables is tight.
  (c) Show that a union of two tight sets is tight.
  (d) Show that any finite set of random variables is tight.

(6) Let $X_1, X_2, \dots$ be a *tight* sequence of random variables.
  (a) Show that there exists a subsequence $(n_k)_{k\in\mathbb{N}}$ and a random variable $X$ such that $X_{n_k} \xrightarrow{D} X$. *Hint: put together previous problems.*
  (b) Assume that $X$ is a random variable which is the "only possible limit of the $X_n$'s", in the sense that for any random variable $Y$, if for a subsequence $(n_k)_k$, $(X_{n_k}) \xrightarrow{D} Y$ then $F_X = F_Y$. Show that $X_n \xrightarrow{D} X$. *Hint: suppose not and use the previous part on an appropriate subsequence.*

(7) Let $X$ be a random variable and let $\phi$ be its characteristic function. Let $T > 0$. Show that:

$$\frac{1}{T}\left|\int_{-T}^{T} \phi(t)dt\right| - 1 \leq P\left(-\frac{2}{T} \leq X \leq \frac{2}{T}\right)$$

*Hint: first show that* $\int_{-T}^{T} \frac{1}{2T} e^{itx} dt = \frac{\sin(Tx)}{Tx}$. *Then fix* $A > 0$ *and integrate separately on* $[-2A, 2A]$ *and* $[-2A, 2A]^c$ *(noting that* $|\frac{\sin(Tx)}{Tx}| \leq 1$ *always and* $\leq \frac{1}{2A}$ *on the second set). Finally, set* $A := \frac{1}{T}$.

(8) Let $X_1, X_2, \dots$ be a sequence of random variables and let $\phi_1, \phi_2, \dots$ be their characteristic function. Let $X$ be a random variable and let $\phi$ be its characteristic function.
  (a) Suppose that $\phi_n \to \phi$ pointwise. Show that $X_1, X_2, \dots$ is a *tight* sequence. *Hint: use the previous problem and continuity of* $\phi$.
  (b) Prove (2) implies (1) in the statement of the continuity theorem (Fact 5.9 in the notes). *Hint: first show that if a subsequence* $X_{n_k}$ *converges to some* $Y$ *in distribution, then* $F_X = F_Y$ *(use (1) implies (2) and the reconstruction theorem). Now use a previous exercise.*