MATH 154 - PROBABILITY THEORY, SPRING 2018
ASSIGNMENT 10

Due Wednesday, April 11 at the beginning of class. Make sure to include your full name *and the list of your collaborators* (if any) with your assignment. You may discuss problems with others, but you may *not* keep a written record of your discussions. Please refer to the syllabus for details.

With regards to answering these problems, imagine that you are writing an answer to teach someone else in the class how to do the problem. In particular, you must give a complete outline for how you arrived at your answer. It is not sufficient to simply state a number or formula without providing the steps and reasoning that you used to produce the answer.

IMPORTANT: *Unless explicitly mentioned,* "random variable" *below refers to Definition 2.1.3 in Grimmett-Stirzaker. That is, random variables are not necessarily discrete or continuous random variables.*

(1) Let X be a discrete or continuous random variable. Let $m < n$ be positive natural numbers. Assume that $\mathbb{E}(|X|^n) < \infty$. Prove that $\mathbb{E}(|X|^m) < \infty$.

(2) A set A of real numbers is called *dense* if for any two real numbers $x < y$, there exists $a \in A$ such that $x < a < y$. For example, $\mathbb{R}$ is dense, $\mathbb{Q}$ is dense (between any two real numbers, there is a rational), but $[0, 1]$ is not dense.
   (a) Show that if B is any countable subset of $\mathbb{R}$, $\mathbb{R}\backslash B$ is dense. *Hint: you may use without proof that any non-empty open interval of real numbers is uncountable.*
   (b) Let X and Y be random variables and let A be a dense set of reals. Assume that $F_X \restriction A = F_Y \restriction A$ (that is, $F_X(x) = F_Y(x)$ for any $x \in A$). Show that $F_X = F_Y$. *Hint: remember that any distribution is right-continuous.*
   (c) Let X and Y be random variables. Assume that $F_X(x) = F_Y(x)$ except for countably-many x. Deduce from the previous parts that $F_X = F_Y$.

(3) Let X and Y be random variables. Show that there are only countably-many x such that $P(X = x) > 0$. *Hint: Suppose not. Find a rational number $q > 0$ and an infinite set A such that for any $x \in A$, $P(X = x) \geq q$. Explain why this yields a contradiction. To find q and A, you may use without proofs the following facts:*
   (a) *The rationals are countable and dense (see the previous problem).*
   (b) *The pigeonhole principle for infinite sets: if $f : B \to C$ is a function, B is uncountable, and C is countable, then there exists an uncountable subset B_0 of B and an element c of C such that $f(x) = c$ for all $x \in B_0$.*

(4) Prove the following basic properties of convergence in distribution. Everywhere below, $X, Y, X_1, X_2, \ldots$ are random variables.
   (a) If $X, X_1, X_2, \ldots$ have the same distribution function F, then $X_n \xrightarrow{D} X$.

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(b) If $Y_1, Y_2, \dots Y_N$ are random variables, then the sequence $Y_1, Y_2, \dots, Y_N, X_1, X_2, \dots$ converges in distribution to X if and only if $X_1, X_2, \dots$ converges in distribution to X.
(c) If $X_n \xrightarrow{D} X$ and $X_n \xrightarrow{D} Y$, then $F_X = F_Y$. *Hint: a countable union of countable sets is countable.*
(d) If $X_n \xrightarrow{D} X$ and $(X_{n_k})_{k \in \mathbb{N}}$ is a subsequence of $X_1, X_2, \dots$, then $X_{n_k} \xrightarrow{D} X$.
(e) If $X_n \xrightarrow{D} X$ and μ is a real number, then $X_n + \mu \xrightarrow{D} X + \mu$.

(5) Let $X_1, X_2, \dots$ be a sequence of discrete or continuous random variables and let μ be a real number. Let $\mu_n := \mathbb{E}(X_n)$ and let $\sigma_n^2 := \mathrm{Var}(X_n)$ (we allow $\sigma_n = \infty$). Show that the following four statements are equivalent:
(a) $X_n \xrightarrow{D} \mu$.
(b) For any $\epsilon > 0$, $P(|X_n - \mu| < \epsilon) \to 1$.
(c) For any $\epsilon > 0$, $P(\exists n \forall m \geq n : |X_m - \mu| < \epsilon) = 1$.
(d) $\mu_n \to \mu$ and $\sigma_n \to 0$.

Deduce short proofs of the law of averages (2.2.1 in Grimmett-Stirzaker), the probability of extinction being 1 for branching process with $\mathbb{E}(Z_1) < 1$ (part of 5.4.5 in Grimmett-Stirzaker), and the law of large numbers when the random variables have finite variances (Theorem 4.7 in the online notes on the limit theorems).

(6) (a) Prove that the characteristic function of a geometric random variable X with parameter p, $0 < p < 1$, is $\phi_X(t) = \frac{pe^{it}}{1-(1-p)e^{it}}$.
(b) Prove that the characteristic function of a Poisson random variable X with parameter λ is $\phi_X(t) = e^{\lambda(e^{it}-1)}$.

(7) Do problem 24 on p. 209 of Grimmett-Stirzaker (X has *Cauchy distribution* if $f_X(x) = \frac{1}{\pi(1+x^2)}$ for $x \in \mathbb{R}$. You may use without proof that the characteristic function of the Cauchy distribution is $\phi_X(t) = e^{-|t|}$; see also the second extra credit problem).

(8) Do problem 32 on p. 210 of Grimmett-Stirzaker.

Extra credit

(1) Prove that the characteristic function of an exponential random variable X with parameter λ is $\phi_X(t) = \frac{\lambda}{\lambda - it}$. *Hint: do NOT treat $e^{(it-\lambda)x}$ in the integral as if it were a real-valued function. Instead, split the integral into its real and imaginary parts (or use contour integration, if you know it). You may use without proof that an antiderivative of $e^{cx}\sin(bx)$ (for b, c real numbers with $bc \neq 0$) is $\frac{e^{cx}}{c^2+b^2}(c\sin(bx) - b\cos(bx))$ and an antiderivative of $e^{cx}\cos(bx)$ is $\frac{e^{cx}}{c^2+b^2}(c\cos(bx) - b\sin(bx))$.*

(2) Prove that the characteristic function of a Cauchy random variable X is $\phi_X(t) = e^{-|t|}$. You may use without proof the Fourier inversion formula: if ϕ is the characteristic function of a random variable X and ϕ is absolutely integrable (i.e. $\int_{-\infty}^{\infty} |\phi(t)| dt < \infty$), then X is continuous and $f_X(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ixt}\phi(t) dt$.