# MATH 154 - PROBABILITY THEORY, SPRING 2018
# ASSIGNMENT 6

**Due Friday, March 9 at the beginning of class.** Make sure to include your full name *and the list of your collaborators* (if any) with your assignment. You may discuss problems with others, but you may *not* keep a written record of your discussions. Please refer to the syllabus for details.

With regards to answering these problems, imagine that you are writing an answer to teach someone else in the class how to do the problem. In particular, you must give a complete outline for how you arrived at your answer. It is not sufficient to simply state a number or formula without providing the steps and reasoning that you used to produce the answer.

(1) Prove that any $\sigma$-field is a semi-algebra.

(2) Let $\mathcal{A}$ be the set of all subintervals of $\Omega = [0, 1]$ (in the sense of the online notes on building probability spaces). Show that $\mathcal{A}$ is a semi-algebra.

(3) Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be semi-algebras on the sample spaces $\Omega_1$ and $\Omega_2$ respectively. Show that $\{A_1 \times A_2 \mid A_1 \in \mathcal{A}_1, A_2 \in \mathcal{A}_2\}$ is a semi-algebra on the sample space $\Omega_1 \times \Omega_2$. *Hint: first show it when $\mathcal{A}_1$ and $\mathcal{A}_2$ are $\sigma$-fields, then expand the proof further.*

(4) Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let $A \in \mathcal{F}$ be such that $P(A) > 0$. We define the *subspace induced by $A$*, a triple $(\Omega^*, \mathcal{F}^*, P^*)$, as follows:
- $\Omega^* = A$.
- $\mathcal{F}^* = \{B \in \mathcal{F} \mid B \subseteq A\}$.
- $P^* : \mathcal{F}^* \to [0, 1]$ is $P^*(B) = P(B|A)$.

Intuitively, this is the subspace obtained when we restrict ourselves only to the outcomes when the event $A$ happens. Prove that $(\Omega^*, \mathcal{F}^*, P^*)$ is a probability space. *Note: This involves checking both that $\mathcal{F}^*$ is a $\sigma$-field and that $P^*$ is a probability function.*

(5) Let $(\Omega, \mathcal{F}, P)$ be a probability space. We call a set $A \in \mathcal{F}$ *null* if $P(A) = 0$. In this problem, you will construct a $\sigma$-field $\mathcal{F}^*$ and a probability function $P^*$ such that $\mathcal{F} \subseteq \mathcal{F}^*$ and $\mathcal{F}^*$ contains all subsets of null events (they will also have probability zero). The space $(\Omega, \mathcal{F}^*, P^*)$ is called the *completion* of $(\Omega, \mathcal{F}, P)$, see 1.6 in Grimmett-Stirzaker.

(a) Let $\mathcal{N} := \{A \subseteq \Omega \mid A \text{ is contained inside a null set}\}$. Show that $\mathcal{N}$ contains the empty set, is closed under taking subsets, and is closed under countable unions.

(b) For sets $A$ and $B$, let $A \Delta B$ denote the *symmetric difference* of $A$ and $B$. It is defined by $A \Delta B = (A \backslash B) \cup (B \backslash A)$. Let $\mathcal{F}^* := \{A \subseteq \Omega \mid A \Delta B \in \mathcal{N} \text{ for some } B \in \mathcal{F}\}$. Prove that $\mathcal{F} \subseteq \mathcal{F}^*$ and $\mathcal{F}^*$ is a $\sigma$-field. *Hint: First draw a picture to understand the definition.*

(c) Let $A \subseteq \Omega$ and let $B_1, B_2 \in \mathcal{F}$. Show that if $A \Delta B_1 \in \mathcal{N}$ and $A \Delta B_2 \in \mathcal{N}$, then $B_1 \Delta B_2 \in \mathcal{N}$. *Hint: First draw a picture to understand this. Then write $B_\ell$ as $(A \cap B_\ell) \cup (A^c \cap B_\ell)$, for $\ell = 1, 2$.*

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*Date*: February 28, 2018.

(d) Define $P^* : \mathcal{F}^* \to [0,1]$ by $P^*(A) = P(B)$ whenever $B \in \mathcal{F}$ is such that $A \Delta B \in \mathcal{N}$. Prove that $P^*$ is well-defined (that is, we get the same result regardless of the choice of $B$) and $P^*$ extends $P$. *Hint: first show that if $B_1, B_2 \in \mathcal{F}$ are such that $B_1 \Delta B_2 \in \mathcal{N}$, then $P(B_1) = P(B_2)$. Then use the previous part.*

## Extra credit problems

(1) In this problem, you will construct a probability space modeling the experiment of tossing infinitely-many fair coins[1]. Let $\mathbb{N} = \{0, 1, 2, \dots\}$ be the set of natural numbers and let $\Omega$ be the set of all functions $f : \mathbb{N} \to \{0, 1\}$. We think of each such function as an infinite sequence of zeroes and ones. For $n$ a natural number, we let $[n] := \{m \in \mathbb{N} \mid m < n\}$. Let $S$ be the set of all functions of the form $f : [n] \to \{0, 1\}$ (a finite sequence of zeroes and ones), for some natural number $n$. For $s \in S$ with domain $[n]$, we say that $f \in \Omega$ *extends* $s$ if $s(m) = f(m)$ for all $m < n$. Let $A_s$ be the set of all $f \in \Omega$ which extend $s$.

(a) Let $\mathcal{A} := \{\emptyset\} \cup \{A_s \mid s \in S\}$. Show that $\mathcal{A}$ is a semi-algebra.

(b) Let $P_0 : \mathcal{A} \to [0,1]$ be defined by $P_0(\emptyset) = 0$ and $P_0(A_s) = \frac{1}{2^n}$, where $n$ is such that $[n]$ is the domain of $s$. Explain why defining $P_0$ this way makes sense if we think of each function $f \in \Omega$ as giving the result of tossing a coin infinitely-many times, with $f(n)$ giving whether the $n$th toss was head or tail.

(c) Show that $P_0$ satisfies the hypotheses of the extension theorem. *Hint: for countable monotonicity, feel free to use any fact from topology that you know. Here is a way to do it if you do not know any topology: show that if $A_s \subseteq \bigcup_{i=1}^{\infty} A_{s_i}$, there must exist a natural number $n$ such that $A_s \subseteq \bigcup_{i=1}^{n} A_{s_i}$. To do this, first reduce to the pairwise disjoint case, then suppose not and define an infinite sequence $f \in \Omega$ which extends $s$ and such that for each $n \in \mathbb{N}$ there are infinitely-many $i$'s with $s_i$ extending the restriction of $f$ to $[n]$. Prove that $f \notin A_s$.*

(2) Let $\mathcal{A}$ be the set of all subintervals of $[0,1]$, and let $P_0 : \mathcal{A} \to [0,1]$ be defined by $P_0([a,b]) = P_0([a,b)) = P_0((a,b]) = P_0((a,b)) = b - a$, for $a \leq b$.

(a) Prove that $P_0$ is finitely additive.

(b) Prove that if $A_1, A_2, \dots$ are all *open intervals* and $A$ is a *closed* interval such that $A \subseteq \bigcup_{i=1}^{\infty} A_i$, then $P_0(A) \leq \sum_{i=1}^{\infty} P_0(A_i)$. *Hint: you may without proof use the Heine-Borel theorem: if the union of a collection of open intervals contains a closed interval, then the union of some finite subcollection of open intervals also contains this closed interval.*

(c) Prove that if $A_1, A_2 \dots$ are arbitrary intervals and $A$ is an arbitrary interval such that $A \subseteq \bigcup_{i=1}^{\infty} A_i$, then $P_0(A) \leq \sum_{i=1}^{\infty} P_0(A_i)$. *Hint: here is one way: grow each $A_i$ to an open interval with $\epsilon 2^{-i}$-more elements padded to each end, then use the previous part.*

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[1]Another approach toward this: identify each real number with its sequence of digits in binary; then Lebesgue measure is essentially the same as what we want to build.