MATH 154 - PROBABILITY THEORY, SPRING 2018
ASSIGNMENT 7

Due Wednesday, March 21 at the beginning of class. Make sure to include your full name *and the list of your collaborators* (if any) with your assignment. You may discuss problems with others, but you may *not* keep a written record of your discussions. Please refer to the syllabus for details.

With regards to answering these problems, imagine that you are writing an answer to teach someone else in the class how to do the problem. In particular, you must give a complete outline for how you arrived at your answer. It is not sufficient to simply state a number or formula without providing the steps and reasoning that you used to produce the answer.

(1) Do problems 1 and 2 on p. 92 of Grimmett-Stirzaker.
(2) Do problem 5 on p. 140 of Grimmett-Stirzaker.
(3) Do problem 10 on p. 141 of Grimmett-Stirzaker.
(4) In this exercise, you will show that the Poisson distribution can be naturally derived from the exponential distribution.
   (a) Let m be a positive integer and λ and t be positive real numbers. Let X be a Gamma random variable with parameters λ and m, and let Z be a Poisson random variable with parameter λt. Show that $P(Z < m) = P(X > t)$.
   (b) Let $X_1, X_2, \ldots$ be a sequence of independent exponential random variables, each with parameter λ. Let $S_n := \sum_{i=1}^n X_i$ (we set $S_0 = 0$). Fix a real number $t > 0$.
      (i) Show that, with probability 1, there exists a natural number n such that $S_n \geq t$.
      (ii) Let $N \geq 0$ be the unique natural number such that $S_N < t$ and $S_{N+1} \geq t$. Show that N is a Poisson random variable with parameter λt. *Hint: To understand what this means, let's say $t = 1$ for simplicity. Think of X_1 as specifying the time until some event occurs, and X_2 as the time that elapses after that until another event occurs, etc. Then N is the number of events that have occured within one unit of time.*
(5) There was an average of 74.8 worldwide aircraft crashes (or, in airline industry jargon, *accidents*) per year during the period 2012-2016[1]. Use this number to model the occurence of aircraft crashes (you may want to use a software to perform the computations) and answer the questions below. Assume that the time that elapses between two aircraft crashes is exponentially distributed, and that these times are independent of one another.

Date: March 9, 2018.

[1]Source: The International Air Transport Association (IATA) `http://www.iata.org/pressroom/pr/Pages/2018-02-22-01.aspx`

(a) How long in average can we expect to wait until the next aircraft crash? Until the next two aircraft crashes? Do these numbers depend on when in the past the last aircraft crash occured?
(b) How likely is is that there are at most 45 crashes in a given year?[2]
(c) What is the probability that six or more aircrafts crash in the month of April?
(d) What is the probability that an aircraft crashes on March 22?

(6) Let X_1, X_2, X_3 be independent and uniformly distributed in $[0,1]$. What is the probability that three rods of length X_1, X_2, and X_3 respectively can be put together to form a triangle? *Hint: you should first convince yourself that three rods of length x, y, and z with $x \leq y \leq z$ can be put together to form a triangle if and only if $z \leq x + y$.*

(7) In this exercise, you will compute $I := \int_0^\infty e^{-x^2}\,dx$.
(a) Prove that I exists. That is, prove that the integral converges.
(b) Show that $I^2 = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)}dxdy$.
(c) Evaluate I^2 using polar coordinates and deduce a value for I.
(d) Conclude that the function $f(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$ is a probability density function. That is, show that $\int_{-\infty}^{\infty} f(x)dx = 1$.

Extra credit problem

Do problem 12 on p. 141 of Grimmett-Stirzaker.

[2]According to IATA, there were only 45 crashes in 2017.