MATH 154 - PROBABILITY THEORY, SPRING 2018
ASSIGNMENT 9

Due Wednesday, April 4 at the beginning of class. Make sure to include your full name *and the list of your collaborators* (if any) with your assignment. You may discuss problems with others, but you may *not* keep a written record of your discussions. Please refer to the syllabus for details.

With regards to answering these problems, imagine that you are writing an answer to teach someone else in the class how to do the problem. In particular, you must give a complete outline for how you arrived at your answer. It is not sufficient to simply state a number or formula without providing the steps and reasoning that you used to produce the answer.

(1) Do problem 2 on p. 170 of Grimmett-Stirzaker.

(2) Fix an integer $d \geq 1$. In a *d-dimensional random walk*, we have d-dimensional random vectors $\vec{X}_1, \vec{X}_2, \ldots$, where for each $i \geq 1$, $\vec{X}_i = (X_i^1, X_i^2, \ldots, X_i^d)$, and $P(X_i^j = 1) = p$, $P(X_i^j = -1) = 1 - p$. We assume that X_i^j is independent of $X_{i'}^{j'}$ whenever either $i \neq i'$ or $j \neq j'$. Let $\vec{S}_0 = \vec{0}$ and $\vec{S}_n = \sum_{i=1}^n \vec{X}_i$ for $n \geq 1$. Assume that $p = 1 - p = \frac{1}{2}$. Prove that the probability of eventual return to the origin (that is, the probability that $S_n = \vec{0}$ for some $n \geq 1$) is 1 if $d = 1$ or $d = 2$ but is strictly less than 1 if $d \geq 3$. *Hint: let $F_{\vec{0}}$ be the generating function of the probabilities $f_{\vec{0}}(n)$ of first return to the origin at time n and let $P_{\vec{0}}$ be the generating function of the probabilities $p_{\vec{0}}(n)$ of returning to the origin at time n. Show that $F_{\vec{0}}(1) = 1$ if and only if $P_{\vec{0}}(1) = \infty$. To estimate $P_{\vec{0}}(1)$, you may use Stirling's bounds:* $\sqrt{2\pi} \cdot n^{n+\frac{1}{2}} e^{-n} \leq n! \leq e \cdot e^{n+\frac{1}{2}} e^{-n}$.

Note : György Pólya (1887-1985) explained the result as follows. Drunken humans, living in two-dimensional space, will tend to always stay around the same places (this can be verified by direct observation), and in particular will always stay near their home. However drunken birds, living in three-dimensional space, would tend to drift farther and farther away and so risk never coming back home again. This explains why birds do not drink alcohol.

(3) Do problems 1 and 5 on p. 175 of Grimmett-Stirzaker.

(4) Use Chebyshev's inequality to prove that the probability of extinction is 1 in a branching process where $\mathbb{E}(Z_1) < 1$.

(5) Fix a real number $a > 0$. Give two different examples of a random variable X such that $P(|X| \geq a) = \frac{\mathbb{E}(X^2)}{a^2}$. *Hint: what are the simplest random variables you know? Note: this shows that the bound given by Chebyshev's inequality cannot in general be improved.*

Date: March 28, 2018.