THE MONTY HALL PROBLEM

You are a contestant at a game show and are asked to choose between one of three doors. Behind one of the door is a car, and behind the two other doors are goats (since you live an urban lifestyle, you want the car). You can have whatever is behind the door you choose to open. You randomly pick a door but before you can open it the host opens one of the other doors, revealing a goat. The host then asks you whether you want to change your choice of door. Should you change?

To model this problem, we assume without loss of generality that you initially picked door number one. The solution is of course symmetric in the other cases. We work on the space $S = [3] \times [3]$ (where $[3]$ abbreviates $\{1, 2, 3\}$), where the first component of the pair gives the door behind which the car is hidden, and the second component is the door that the host opened. We know for example that $P(\{(a, a)\}) = 0$ for all $a \in [3]$, since the host always opens a door that has a goat behind it, and also $P(\{(a, 1)\}) = 0$ since you picked the first door. On the other hand, it is not that easy to figure out the probability of the other outcomes.

For $i \in [3]$, let H_i denote the event that the host selected door i, and let A_i be the event that the car is behind door i. We want to compute $P(A_1|H_2)$ and $P(A_1|H_3)$ to figure out what the chances are that there is a car behind the first door. Let's compute $P(A_1|H_2)$, and the symmetric computation will give us that $P(A_1|H_3) = P(A_1|H_2)$. By Bayes' formula (Exercise 1.4.1 in Grimmett-Stirzaker), $P(A_1|H_2) = \frac{P(H_2|A_1)P(A_1)}{P(H_2)}$. Now, $P(A_1) = \frac{1}{3}$ since the car is equally likely to be behind each door. Also, $P(H_2|A_1) = \frac{1}{2}$: if there is a car behind the first door, then the host can choose to open either the second or the third door and will do so (we assume) with equal probability.

It remains to compute $P(H_2)$. For this, we condition on the pairwise disjoint events A_1, A_2, and A_3 (note that $S = A_1 \cup A_2 \cup A_3$):

$$P(H_2) = P(H_2|A_1)P(A_1) + P(H_2|A_2)P(A_2) + P(H_2|A_3)P(A_3)$$

We have already figured out what $P(H_2|A_1)P(A_1)$ is. We also know that $P(A_i) = \frac{1}{3}$ for all $i \in [3]$. Now if the prize is behind the second door, then since you picked the first door, the host has no choice but to open door 3. Thus $P(H_2|A_2) = 0$. Similarly, $P(H_2|A_3) = 1$. Thus $P(H_2) = \frac{1}{2} \cdot \frac{1}{3} + 0 + 1 \cdot \frac{1}{3} = \frac{1}{2}$. Putting everything together, $P(A_1|H_2) = \frac{1/6}{1/2} = \frac{1}{3}$. Therefore $P(A_3|H_2) = P(A_1^c|H_2) = 1 - P(A_1|H_2) = \frac{2}{3}$ (it is easy to check that the rule for computing the probability of complements also holds when conditioning).

In conclusion, the car is much more likely to be behind the other door and therefore you should accept the host's offer and change your choice of door.