NOTES ON THE "MAD SECRETARY"

A secretary puts n letters randomly in n labelled enveloppe. What is the probability that exactly r letters are put in the correct enveloppe? This problem is considered on p. 56 of Grimmett-Stirzaker, but we give a slightly different approach using the inclusion-exclusion formula.

Mathematically, we want the probability that a random permutation of $\{1, \ldots, n\}$ fixes exactly r-many elements. Let us call this probability $p_{r,n}$. We assume of course that each permutation is equally likely.

First, let A_i be the event that the number i is fixed (i.e. the letter i ends up in the right enveloppe). For $i_1 < i_2 < \ldots < i_r \leq n$, we have that:

$$P(A_{i_1}, \ldots, A_{i_r}) = P(A_1, \ldots, A_r) = \frac{(n-r)!}{n!}$$

This is because the total number of permutations is $n!$, and the number of permutations that fix the numbers 1 up to r is the number of permutations of the numbers $\{r+1, \ldots, n\}$, and there are $n - r$ such numbers.

Second, observe that for a fixed $S \subseteq \{1, \ldots, n\}$ of size r, the probability that exactly the numbers in S are fixed by a random permutation is $P(\bigcap_{i \in S} A_i) p_{0,n-r} = P(A_1, \ldots, A_r) p_{0,n-r}$. Now summing over all S of size r, we obtain that

$$p_{r,n} = \binom{n}{r} \frac{(n-r)!}{n!} p_{0,n-r} = \frac{p_{0,n-r}}{r!}$$

Let us now compute $p_{0,n}$. This is $1 - P(A_1 \cup A_2 \cup \ldots \cup A_n)$. By the inclusion-exclusion formula ((2) on p. 56 of Grimmett-Stirzaker), we have that $P(A_1 \cup A_2 \cup \ldots \cup A_n) = \sum_{\emptyset \neq S \subseteq \{1,\ldots,n\}} (-1)^{|S|+1} P(\bigcap_{i \in S} A_i)$. Since $P(\bigcap_{i \in S} A_i)$ depends only on $|S|$, we obtain the symmetric version of the inclusion-exclusion formula:

$$P(A_1 \cup A_2 \cup \ldots \cup A_n) = \sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} P(A_1, \ldots, A_k)$$

Putting everything together, we get that $p_{0,n} = \sum_{k=0}^{n} \frac{(-1)^k}{k!}$. Thus as $n \to \infty$, $p_{0,n} \to e^{-1}$.