STIRLING'S FORMULA

Stirling's formula says that for a natural number n, $n!$ is approximately equal to $n^n e^{-n}\sqrt{2\pi n}$, in the sense that their ratio tends to 1:

Theorem (Stirling's formula)**.**

$$\lim_{n\to\infty} \frac{n^n e^{-n}\sqrt{2\pi n}}{n!} = 1$$

We give a short proof using the central limit theorem and the theory of characteristic functions. This is based on Example 5.9.6 in Grimmett-Stirzaker. The idea is to study what happens if we apply the central limit theorem to a sequence of exponential random variables. We will use the following result (a proof is sketched in the extra credit assignment).

Fact (Fourier inversion formula)**.** Let X be a random variable and let ϕ be its characteristic function. If $\int_{-\infty}^{\infty} |\phi(t)|dt < \infty$, then X is a continuous random variable and its density function is given by:

$$f_X(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}\phi(t)dt$$

Proof of Stirling's formula. Let $X_1, X_2, \ldots$ be a sequence of independent exponential random variable with parameter $\lambda = 1$. Let $S_n := X_1 + \ldots + X_n$. By a problem in assignment 7, S_n is a Gamma random variable with parameters 1 and n. That is, $f_{S_n}(x) = \frac{1}{\Gamma(n)} x^{n-1}e^{-x}$ for $x \geq 0$. Recall that $\Gamma(n) = (n-1)!$ for any natural number $n \geq 1$. Note also that the characteristic function of an exponential random variable with parameter $\lambda = 1$ is $(1-it)^{-1}$, so $\phi_{S_n}(t) = (1-it)^{-n}$.

The mean and variance of an exponential with parameter $\lambda = 1$ are 1 (exercise), so let us normalize and consider $T_n := \frac{S_n - n}{\sqrt{n}}$. By the central limit theorem, $T_n \xrightarrow{D} N(0,1)$. It is also straightforward to derive from the characteristic function of S_n that $\phi_{T_n}(t) = e^{-i\sqrt{n}t}\left(1 - \frac{it}{\sqrt{n}}\right)^{-n}$. By the continuity theorem, we know that ϕ_{T_n} will converge pointwise to the characteristic function of a $N(0,1)$ random variable as $n \to \infty$. In other words, $\phi_{T_n}(t) \to e^{-t^2/2}$.

Using the change of variable formula (4.7.3 in Grimmett-Stirzaker) we can also directly write down f_{T_n}. For $x \geq -\sqrt{n}$:

$$f_{T_n}(x) = \sqrt{n} f_{S_n}(x\sqrt{n} + n) = \frac{\sqrt{n}(x\sqrt{n}+n)^{n-1}e^{-(x\sqrt{n}+n)}}{\Gamma(n)}$$

In particular,

$$f_{T_n}(0) = \frac{n^{n-1/2}e^{-n}}{\Gamma(n)} = \frac{n^{n-1/2}e^{-n}}{(n-1)!} = \frac{n \cdot n^{n-1/2}e^{-n}}{n!} = \frac{n^n e^{-n}\sqrt{n}}{n!}$$

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To prove Stirling's formula, we need to check that this goes to $\frac{1}{\sqrt{2\pi}}$ as $n \to \infty$. Note that ϕ_{T_n} is absolutely integrable when $n \geq 2$ (because $|\phi_{T_n}(t)|$ behaves like $\frac{1}{|t|^n}$ when t is big – you may also want to draw a picture of the numbers 1 and $\frac{it}{\sqrt{n}}$ on the complex plane). Thus using the Fourier inversion formula, for $n \geq 2$ we have:

$$f_{T_n}(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi_{T_n}(t) dt$$

As $n \to \infty$, we saw that $\phi_{T_n}(t) \to e^{-t^2/2}$. Moreover, there is a function g such that for all $n \geq 2$, $|\phi_{T_n}(t)| \leq |g(t)|$ and $\int_{-\infty}^{\infty} |g(t)| dt < \infty$ (exercise; first prove the simpler result that there is such a g uniformly bounding $\frac{1}{t^n}$, for $t \geq 1$ and $n \geq 2$). Therefore by the dominated convergence theorem,

$$\lim_{n\to\infty} \frac{n^n e^{-n} \sqrt{n}}{n!} = \lim_{n\to\infty} f_{T_n}(0) = \lim_{n\to\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi_{T_n}(t) dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-t^2/2} dt = \frac{1}{\sqrt{2\pi}}$$

as desired. □