Saturation and solvability in abstract elementary classes with amalgamation

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Introduction

Theorem (Morley, 1965)

A countable first-order theory categorical in *some* uncountable cardinal is categorical in *all* uncountable cardinals.

The proof led to the development of *stability theory* which has had a large impact, both inside and outside model theory.

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Shelah conjectured the following generalization to non-elementary classes:

Conjecture (Shelah, 1970's)

An $\mathbb{L}_{\omega_1,\omega}$-sentence categorical in *some* $\lambda \geq \beth_{\omega_1}$ is categorical in *all* $\lambda' \geq \beth_{\omega_1}$.

This has fueled a lot of research, with thousand of pages of approximation, but is still open.

Introduction

A key notion on Morley's proof is that of a saturated model. Part of Morley's proof shows:

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In this talk, we will generalize this step to $\mathbb{L}_{\omega_1,\omega}$ and more generally to AECs.

Abstract elementary classes

An AEC is a pair $\mathbf{K} = (K, \leq_{\mathbf{K}})$, where K is a class of structures in a fixed vocabulary $\tau(\mathbf{K})$ and $\leq_{\mathbf{K}}$ is a partial order on $\mathbf{K}$ satisfying some of the basic category-theoretic properties of $(\mathrm{Mod}(T), \preceq)$.

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For example, $\mathbf{K}$ is closed under unions of $\leq_{\mathbf{K}}$-increasing chains and satisfies the downward Löwenheim-Skolem-Tarski theorem. More precisely:

There exists a (least) cardinal $\mathrm{LS}(\mathbf{K}) \geq |\tau(\mathbf{K})| + \aleph_0$ such that for any $M \in \mathbf{K}$ and any $A \subseteq |M|$, there is $M_0 \leq_{\mathbf{K}} M$ containing A with $\|M_0\| \leq |A| + \mathrm{LS}(\mathbf{K})$.

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Examples include $(\mathrm{Mod}(T), \preceq)$ (where $\mathrm{LS}(\mathbf{K}) = |T|$), $(\mathrm{Mod}(\psi), \preceq_{\Phi})$ (where $\mathrm{LS}(\mathbf{K}) = |\Phi| + |\tau(\Phi)| + \aleph_0$), and more generally classes of models of $\mathbb{L}_{\lambda^+,\omega}(Q)$ sentences.

The monster model

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In this case (imitating the Fraïssé construction) one can build a class-sized model $\mathfrak{C}$. Such that:

1. $\mathfrak{C}$ is *universal*: For $M \in \mathbf{K}$, there is a $\mathbf{K}$-embedding $f : M \to \mathfrak{C}$ (i.e. $f : M \cong f[M]$ and $f[M] \leq_{\mathbf{K}} \mathfrak{C}$).
2. $\mathfrak{C}$ is *model-homogeneous*: For $M \leq_{\mathbf{K}} \mathfrak{C}$ and $M \leq_{\mathbf{K}} N$, there is $f : N \xrightarrow[M]{} \mathfrak{C}$.

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We call $\mathfrak{C}$ the *monster model*. We say that $\mathbf{K}$ *has a monster model* if it has amalgamation, joint embedding, and arbitrarily large models.

Types

From now on, fix an AEC $\mathbf{K}$ with a monster model. Assume every object we work with lives inside the monster model.

Definition

Let $\mathbf{gtp}(a/M)$ (the *Galois* type of a over M) be the orbit of a under automorphisms of $\mathfrak{C}$ fixing M. Naturally define what it means to realize a type, restrict a type, etc.

Saturation and homogeneity

Let $\lambda > \text{LS}(\mathbf{K})$ and let $M \in \mathbf{K}_{\geq \lambda}$.

Definition

1. M is λ-*saturated* if for any $M_0 \in \mathbf{K}_{<\lambda}$ with $M_0 \leq_{\mathbf{K}} M$, any (Galois) type over M_0 is realized in M.
2. M is λ-*model-homogeneous* if for any $M_0 \in \mathbf{K}_{<\lambda}$ with $M_0 \leq_{\mathbf{K}} M$, M is universal over M_0 (i.e. any $M_0' \geq M_0$ with $\|M_0'\| = \|M_0\|$ embeds into M over M_0).

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Lemma ("model-homogeneous = saturated", Shelah)

M is λ-model-homogeneous if and only if M is λ-saturated. In particular, there is at most one saturated model of a given size.

Main theorem

Theorem (V.)

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Note: Morley's proof shows that **K** is stable in λ. However there is an example (Hart-Shelah, Baldwin-Kolesnikov) of an $\mathbb{L}_{\omega_1,\omega}$-sentence with a monster model categorical in $\aleph_0, \ldots, \aleph_n$ but unstable in $\aleph_n$ (hence not categorical in $\aleph_{n+1}$).

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The real goal behind solving such questions is to develop a *superstability theory for AECs*.

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Another application of the superstability theory of AECs:

Theorem (V.)

A *universal* $\mathbb{L}_{\omega_1,\omega}$ sentence that is categorical in *some* $\lambda \geq \beth_{\beth_{\omega_1}}$ is categorical in *all* $\lambda' \geq \beth_{\beth_{\omega_1}}$.

Splitting-like independence

Definition (Shelah)

For $M \leq_{\mathbf{K}} N$, $p \in \mathrm{gS}(N)$ *λ-splits over M* if there exists $N_1, N_2 \in \mathbf{K}_\lambda$ such that $M \leq_{\mathbf{K}} N_\ell \leq_{\mathbf{K}} N$ for $\ell = 1, 2$ and $f : N_1 \cong_M N_2$ such that $f(p \upharpoonright N_1) \neq p \upharpoonright N_2$.

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Definition

An AEC $\mathbf{K}$ (with a monster model) is *λ-superstable* if $\lambda \geq \mathrm{LS}(\mathbf{K})$, $\mathbf{K}$ is stable in λ, and $\mathbf{K}$ has *no long splitting chains in λ*: for any $\delta < \lambda^+$, any $\langle M_i : i \leq \delta \rangle$ increasing continuous with M_{i+1} universal over M_i, any $p \in \mathrm{gS}(M_\delta)$, there exists $i < \delta$ such that p does not λ-split over M_i.

It turns out that for a first-order T, T is λ-superstable if and only if T is superstable and stable in λ.

Limit models

By the "model-homogeneous = saturated" lemma, any two saturated models are isomorphic.

Sometimes, we will want to work in a single cardinal only. We attempt to replace saturated models with *limit models*:

Definition (Shelah)

Let $\mathbf{K}$ be an AEC with a monster model. Let $\lambda \geq \text{LS}(\mathbf{K})$ be such that $\mathbf{K}$ is stable in λ. Let $M_0 \leq_{\mathbf{K}} M$ both be in $\mathbf{K}_\lambda$ and let δ be a limit ordinal. We say that M is *(λ, δ)-limit over M_0* if there exists $\langle N_i : i \leq \delta \rangle$ increasing continuous with $M_0 = N_0$, $M = N_\delta$, and N_{i+1} universal over N_i for all $i < \delta$.

Uniqueness of limit models

Question

If M_1, M_2 are respectively (λ, δ_1), (λ, δ_2)-limit over M_0, do we have that $M_1 \cong_{M_0} M_2$?

The answer is yes if $\text{cf}(\delta_1) = \text{cf}(\delta_2)$ (do a back and forth argument).

If the answer is yes, then the limit model will be saturated (when $\lambda > \text{LS}(\mathbf{K})$).

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Uniqueness of limit models is closely related to unions of chains of λ-saturated models being λ-saturated.

For T a first-order theory, limit models are unique if and only if T is superstable. If T is stable, limit models of length at least $\kappa_r(T)$ will be isomorphic.

When is an AEC superstable?

Theorem (Shelah-Villaveces)

Let $\lambda \geq \mathrm{LS}(\mathbf{K})$. If $\mathbf{K}$ is categorical in some cardinal strictly above λ, then $\mathbf{K}$ is λ-superstable.

There are some other criterias involving tameness (e.g. in this case, stability on a tail implies superstability).

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If **K** is λ-superstable and splitting has λ-symmetry, then limit models of cardinality λ are unique.

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If symmetry fails, then one can build a sequence $\langle \bar{a}_i : i < \lambda^+ \rangle$ witnessing a certain order property (this comes from a joint paper with Monica VanDieren).

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Using categoricity, one then embeds this sequence inside the Ehrenfeucht-Mostowski model generated by λ^+. One can then use a Δ-system argument (due to Shelah) to get an "EM-indiscernible" subsequence $\langle \bar{a}_i : i \in I \rangle$ with $I \subseteq \lambda^+$ of size λ^+.

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This sequence can be extended to any arbitrary linear order, hence will generate many types, contradicting stability below the categoricity cardinal. □

Solvability

Definition (Superlimit, Shelah)

$M \in \mathbf{K}_\lambda$ is *superlimit* if it is universal in $\mathbf{K}_\lambda$ and whenever $\langle M_i : i \leq \delta \rangle$ is increasing continuous with $M_i \cong M$ for all $i < \delta < \lambda^+$, then $M_\delta \cong M$.

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Definition (Solvability, Shelah)

$\mathbf{K}$ is λ-*solvable* if there is an blueprint Φ of size $\mathrm{LS}(\mathbf{K})$ such that for every linear order I of size λ, $\mathrm{EM}_\tau(I, \Phi)$ is superlimit.

The proof of the main theorem generalizes to show that the superlimit model of size λ is saturated in case $\mathbf{K}$ is λ-solvable.

Shelah's eventual solvability conjecture

Conjecture (Shelah)

If $\mathbf{K}$ is solvable in *some* high-enough cardinal, then (for some μ), $\mathbf{K}_{\geq \mu}$ is solvable in *all* high-enough cardinals.

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Theorem (V.)

If **K** has a monster model, solvability transfers down.

Theorem (Grossberg-V.)

A first-order theory T is solvable (in some $\lambda > |T|$) if and only if it is stable below λ and superstable. In fact, if **K** is an $\text{LS}(\mathbf{K})$-tame AEC with a monster model, solvability in *some* $\lambda > \text{LS}(\mathbf{K})$ implies solvability in *all* $\lambda' \geq \beth_{\omega+\omega}(\text{LS}(\mathbf{K}))$.

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