Superstability in abstract elementary classes

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Introduction

Theorem (Shelah)

Let T be a first-order theory. The following are equivalent:

1. T is stable in all $\lambda \geq 2^{|T|}$.
2. T is stable and $\kappa(T) = \aleph_0$.
3. T has a saturated model in every cardinal $\lambda \geq 2^{|T|}$.

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Theorem (Shelah)

Let T be a stable first-order theory and let $2^{|T|} \leq \lambda < \lambda^{<\lambda}$. The following are equivalent:

1. T is stable in λ.
2. $\lambda = \lambda^{<\kappa(T)}$.
3. T has a saturated model of cardinality λ.

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2. The behavior of forking.
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Question

Can we generalize these results to non-elementary contexts?

Why would we want to do that? To apply the theory to more examples and better understand first-order superstability.

Two applications

Theorem (V.)

Let ψ be a universal $\mathbb{L}_{\omega_1,\omega}$-sentence. If ψ is categorical in *some* $\lambda \geq \beth_{\beth_{\omega_1}}$, then ψ is categorical in *all* $\lambda' \geq \beth_{\beth_{\omega_1}}$.

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Theorem (V.)

Let **K** be an AEC with a monster model. Let $\lambda > \text{LS}(\mathbf{K})$. If **K** is categorical in λ, then the model of cardinality λ is (Galois) saturated.

Saturation and homogeneity

From now on, assume that $\mathbf{K}$ is an AEC with a monster model. Let $\lambda > \mathrm{LS}(\mathbf{K})$ and let $M \in \mathbf{K}_{\geq \lambda}$.

Definition

1. M is *λ-saturated* if for any $M_0 \in \mathbf{K}_{<\lambda}$ with $M_0 \leq_{\mathbf{K}} M$, any (Galois) type over M_0 is realized in M.
2. M is *λ-model-homogeneous* if for any $M_0 \in \mathbf{K}_{<\lambda}$ with $M_0 \leq_{\mathbf{K}} M$, M is universal over M_0 (i.e. any $M_0' \geq M_0$ with $\|M_0'\| = \|M_0\|$ embeds into M over M_0).

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Lemma ("model-homogeneous = saturated", Shelah)

1. M is λ-model-homogeneous if and only if M is λ-saturated.
2. If $\mathbf{K}$ is stable in μ and $M \in \mathbf{K}_\mu$, then there exists $N \in \mathbf{K}_\mu$ with N universal over M.

Limit models

By the "model-homogeneous = saturated" lemma, any two saturated models are isomorphic.

Sometimes, we will want to work in a single cardinal only. We attempt to replace saturated models with *limit models*:

Definition (Shelah)

Let $\mathbf{K}$ be an AEC with a monster model. Let $\lambda \geq \text{LS}(\mathbf{K})$ be such that $\mathbf{K}$ is stable in λ. Let $M_0 \leq_{\mathbf{K}} M$ both be in $\mathbf{K}_\lambda$ and let δ be a limit ordinal. We say that M is *(λ, δ)-limit over M_0* if there exists $\langle N_i : i \leq \delta \rangle$ increasing continuous with $M_0 = N_0$, $M = N_\delta$, and N_{i+1} universal over N_i for all $i < \delta$.

Uniqueness of limit models

Question

If M_1, M_2 are respectively (λ, δ_1), (λ, δ_2)-limit over M_0, do we have that $M_1 \cong_{M_0} M_2$?

The answer is yes if $\text{cf}(\delta_1) = \text{cf}(\delta_2)$ (do a back and forth argument).

If the answer is yes, then the limit model will be saturated (when $\lambda > \text{LS}(\mathbf{K})$).

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If the answer is yes, then the limit model will be saturated (when $\lambda > \text{LS}(\mathbf{K})$).

Uniqueness of limit models is closely related to unions of chains of λ-saturated models being λ-saturated.

For T a first-order theory, limit models are unique if and only if T is superstable. If T is stable, limit models of length at least $\kappa_r(T)$ will be isomorphic.

Splitting-like independence

Definition (Shelah)

For $M \leq_{\mathbf{K}} N$, $p \in \mathrm{gS}(N)$ *λ-splits over M* if there exists $N_1, N_2 \in \mathbf{K}_\lambda$ such that $M \leq_{\mathbf{K}} N_\ell \leq_{\mathbf{K}} N$ for $\ell = 1, 2$ and $f : N_1 \cong_M N_2$ such that $f(p \restriction N_1) \neq p \restriction N_2$.

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Definition

An AEC $\mathbf{K}$ (with a monster model) is *λ-superstable* if $\lambda \geq \mathrm{LS}(\mathbf{K})$, $\mathbf{K}$ is stable in λ, and $\mathbf{K}$ has *no long splitting chains in λ*: for any $\delta < \lambda^+$, any $\langle M_i : i \leq \delta \rangle$ increasing continuous with M_{i+1} universal over M_i, any $p \in \mathrm{gS}(M_\delta)$, there exists $i < \delta$ such that p does not λ-split over M_i.

It turns out that for a first-order T, T is λ-superstable if and only if T is superstable and stable in λ.

Forking-like independence

Definition (V.)

For $M \leq_{\mathbf{K}} N$ both in $\mathbf{K}_\lambda$, $p \in \mathrm{gS}(N)$ *does not λ-fork over M* if there exists $M_0 \in \mathbf{K}_\lambda$ such that M is universal over M_0 and p does not λ-split over M_0.

Assuming λ-superstability, λ-nonforking is well-behaved over limit models: types have unique nonforking extensions.

When is an AEC superstable?

Theorem (Shelah-Villaveces)

Let $\lambda \geq \mathrm{LS}(\mathbf{K})$. If $\mathbf{K}$ is categorical in some cardinal strictly above λ, then $\mathbf{K}$ is λ-superstable.

When is an AEC superstable?

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Let $\lambda \geq \text{LS}(\mathbf{K})$. If $\mathbf{K}$ is categorical in some cardinal strictly above λ, then $\mathbf{K}$ is λ-superstable.

Theorem (V.)

Let $\lambda > \mu \geq \text{LS}(\mathbf{K})$. If $\mathbf{K}$ is stable in λ, μ-tame, and has a unique limit model of cardinality λ, then $\mathbf{K}$ is λ-superstable.

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Theorem (V.)

If $\mathbf{K}$ is λ-superstable and λ-tame, then $\mathbf{K}$ is λ'-superstable for all $\lambda' \geq \lambda$. In this case, λ-nonforking "transfers up" and becomes well-behaved for types over λ^+-saturated models.

When is an AEC superstable?

Theorem (V.)

If **K** is μ-tame and stable in all $\theta \in [\mu, \beth_{(2^\mu)^+})$, then **K** is $\beth_{(2^\mu)^+}$-superstable.

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More generally, one can (assuming SCH) characterize the eventual stability spectrum of tame AECs:

Theorem (V.)

Assume SCH. Let **K** be a μ-tame AEC that is stable in some cardinal above μ. There exists a cardinal $\lambda'(\mathbf{K})$ and a class $\underline{\chi}(\mathbf{K})$ of regular cardinals such that:

1. If $\theta \geq \beth_{(2^\mu)^+}$ is regular, then $\theta \in \underline{\chi}(\mathbf{K})$.
2. For all $\lambda \geq \lambda'(\mathbf{K})$, **K** is stable in λ if and only if $\text{cf}(\lambda) \in \underline{\chi}(\mathbf{K})$.

When does superstability imply the uniqueness of limit models?

Question

If $\mathbf{K}$ is λ-superstable, are limit models of cardinality λ unique?

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Definition

$\mathbf{K}$ has λ-*symmetry* if the following are equivalent for $M \in \mathbf{K}_\lambda$ limit, $a, b \in \mathfrak{C}$.

1. There exists $M_b \in \mathbf{K}_\lambda$ containing b with $M \leq_{\mathbf{K}} M_b$ such that $\mathbf{tp}(a/M_b)$ does not λ-fork over M.
2. There exists $M_a \in \mathbf{K}_\lambda$ containing a with $M \leq_{\mathbf{K}} M_a$ such that $\mathbf{tp}(b/M_a)$ does not λ-fork over M.

When does superstability imply the uniqueness of limit models?

Theorem (VanDieren)

If **K** is λ-superstable and has λ-symmetry, then limit models of cardinality λ are unique.

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Theorem (VanDieren-V.)

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Theorem (VanDieren-V.)

If **K** is λ'-superstable for all $\lambda' \geq \lambda$, then **K** has λ-symmetry.

Theorem (V.)

Let $\lambda \geq \mathrm{LS}(\mathbf{K})$. If **K** is categorical in some cardinal strictly above λ, then **K** has λ-symmetry.

Uniqueness of limit models in strictly stable AECs

Theorem (Boney-V.)

If **K** is stable and μ-tame, then for any stability cardinal $\lambda \geq \beth_{(2^\mu)^+}$, unions of chains of λ-saturated models of cofinality at least $\beth_{(2^\mu)^+}$ are λ-saturated.

Theorem (Boney-VanDieren)

If **K** is stable in λ and λ-splitting has a continuity property, then limit models of length at least χ are unique (where χ is the least regular such that **K** has no long splitting chains of length $\geq \chi$).

Putting it all together

Theorem

Let **K** be a μ-tame AEC stable in some cardinal above μ. The following are equivalent:

1. **K** is stable on a tail of cardinals.
2. **K** has no long splitting chains in all high-enough cardinals.
3. **K** has a unique limit models in all high-enough cardinals.
4. **K** has a saturated model in all high-enough cardinals.

(3 implies 2 was first proven in a joint paper with Rami Grossberg).

Assuming SCH, there is a (more complicated to state) analog to strictly stable AECs.

Some open questions

Theorem (V.)

If $\mathbf{K}$ is a μ-tame AEC stable on some cardinal above μ, then there is a stability cardinal below $\beth_{(2^\mu)^+}$.

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Let **K** be a μ-tame AEC stable on a tail of cardinals. Is there a reasonable bound on the least λ such that **K** is λ-superstable?

If the AEC is $(< \aleph_0)$-tame), the least superstability cardinal is known to be below $\beth_{\beth_{(2^\mu)^+}}$.

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Question

Is there a (ZFC) characterization of the stability spectrum of tame AECs?

References

- Will Boney and Sebastien Vasey, *Chains of saturated models in AECs*, Preprint.
- Rami Grossberg and Sebastien Vasey, *Equivalent definitions of superstability in abstract elementary classes*. To appear in the JSL.
- Monica VanDieren and Sebastien Vasey. *Symmetry in abstract elementary classes with amalgamation*. Preprint.
- Sebastien Vasey, *Saturation and solvability in abstract elementary classes with amalgamation*. Preprint.
- Sebastien Vasey, *Toward a stability theory of tame abstract elementary classes*. Preprint.
- Will Boney, Rami Grossberg, Monica VanDieren, and Sebastien Vasey, *Superstability from categoricity in abstract elementary classes*. To appear in APAL.